MathPath Breakout Catalog 2020
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Week 1, Morning

Elementary Graph Theory, Dr. Jane (Jane Butterfield)

Most people think of a “graph” as a visual representation of data - a function, assorted information, etc. In the study of graph theory, we define a “graph” to be a set of points, called “vertices,” and a set of lines connecting two of those “vertices,” called “edges.” Graphs of this sort can be used to diagram, understand, and solve many mathematical problems; some of which may surprise you! In this breakout, we will be investigating the fundamentals of graph theory, and discovering problems that can be solved by being diagrammed as graphs. *1 star*

Number Theory I, Prof B (Owen Byer)

Number theory has delighted young mathematicians throughout the years due to the accessibility of its ideas and the ingenuity of its techniques. This introductory level breakout presents the foundations of the subject, starting with divisibility and moving on to the Euclidean Algorithm, primes, factoring, counting divisors, perfect numbers, and relatively prime integers. We will conclude with a first look at congruences and modular arithmetic. *1 star*

Wallpaper Patterns and Life on the Klein Bottle, Prof Cahn (Patricia Cahn)

What is it like to live on a Donut? A Mobius Strip? A Klein Bottle? We'll think about this using wallpaper patterns—repeating patterns in the plane with nice symmetries. We'll learn how mathematicians classify these patterns, and practice telling two wallpaper patterns apart with a quick glance. Then we'll fold the patterns up to produce lots of interesting surfaces, called orbifolds, and study life in these spaces. There are no specific prerequisites for this class. *3 stars*

Elementary Logic, Prof D (Matt DeLong)

If you attend MathPath, you cannot be a cow. Bessie is a cow. Most of us would correctly deduce that Bessie cannot attend MathPath, but how do we know this, and how can we prove it? Logic serves as the foundation for our reasoning, setting out formally the rules we understand intuitively. Much as algebra allows us to generalize relationships among numbers by using variables to represent quantities, we will use symbols to represent propositions, demonstrating the form or structure of an argument (premises leading to a conclusion). Then we will formally prove these arguments using rules such as Modus Ponendo Ponens, Modus Tollendo Tollens, and Reductio Ad Absurdum. We will explore truth tables, semantic trees, symbolic logic proofs, and potentially predicate logic. Also, there are paradoxes which seem to defy common sense. We shall see how putting some paradoxical examples into the regimented forms demanded by logic causes the paradoxical element to disappear. *1-2 stars*

Special Relativity: The Mathematics of Paradox, Silas (Silas Johnson)

Einstein's Special Theory of Relativity, at heart, is not about physics so much as the geometry of the 4-dimensional space-time we live in. Motivated by the paradoxes that plague older ways of thinking about physics, we'll discover the equations that define this geometry. We'll finish by using these all-important equations to resolve the paradoxes we explored earlier in the week. If we have extra time, we might take a look at why faster-than-light travel is impossible. *3 stars*

Origami, Asia (Asia Matthews)

The mathematical worlds that we choose to create and explore sometimes lead to surprising results. For example, if our world is made up of only circles and lines, the Greeks showed that while we can draw and bisect any angle we want (is this even true?), it is NOT possible to trisect an arbitrary angle. But strangely, if we begin with a square piece of paper, it IS possible to construct a trisection of an arbitrary angle. What is going on here? What do we have with the square piece of paper that we do not have with lines and circles? In this breakout session, we will play with paper folding to investigate this problem and other geometric results, we will fold checkerboards, discuss the fold-and-cut problem (proved less than 20 years ago) and we will discuss the intersection of mathematics and art. Come ready to apply yourself to the logical deductive process. *2 stars*

Week 1, Afternoon

Basic Counting (Combinatorics), Dr. Jane (Jane Butterfield)

For students who know a little about how to count, but want to know more and get better. We start with the basics: sum rule, product rule, permutations, combinations, the binomial theorem. Then we learn about “combinatorial arguments”, sometimes called proof by story. Time permitting, we will look at inclusion-exclusion, and pigeon-hole arguments. This course is a good prerequisite for various later courses that involve counting. *1 star*

Induction, Prof B (Owen Byer)

If you line up infinitely many dominoes on their ends, with each one close enough to the previous one, and knock over the first, then all infinitely many fall down. This is the essence of mathematical induction, the main proof technique when you have infinitely many statements to prove indexed by the integers, such as “1-ring Towers of Hanoi can be won. 2-ring Towers of Hanoi can be won…. 487-ring Towers of Hanoi can be won….”

Every budding mathematician needs to know mathematical induction, and it's a great proof technique to learn first, because it has a standard template (unlike most proof techniques) and yet leaves room for an infinite amount of variety and ingenuity. Thus this course is one of MathPath's foundation courses.

But don’t take my word for it. Here is what a MathPath student wrote on an AoPS MathPath forum:

> I took induction last year, and I knew induction before the class. But it was very well taught, and I learned how to write proofs by induction, which was very valuable on the USAJMO. I'd recommend this class to anyone who wants to learn about induction, whether you know it at the beginning or not.

The point: Even if you have done induction before, you don't really *know* induction, because it can be used in so many ways in so many parts of mathematics. Every mathematician will probably do 1000 inductions in his/her life. Get 50 under your belt in this course. *2 stars*

To Infinity and Beyond, Dr. Hill (Aaron Hill)

Which set is larger: the set of positive integers {1,2,3,4,…} or the set of all integers {…,-2,-1,0,1,2,…}? Does it make sense to compute the sum 1/1+1/2+1/3+1/4+…? Is it possible to completely cover the plane with non-overlapping segments? These sorts of questions, drawn from such distinct branches of mathematics as set theory, analysis, and geometry, all force us to confront the notion of infinity and the often counterintuitive phenomena that arise once infinite sets or sums are considered. The ideas and mathematical tools that we will need for this breakout should be accessible to any student with a basic background in algebra; our goal will be to enjoy and explore various topics in which infinity crops up and makes life interesting. *2 stars*

Complex numbers, Asia (Asia Matthews)

Complex numbers are just (linear) combinations of real and imaginary numbers! First, I'll tell you a story about how imaginary numbers arose naturally from studying geometry… and then algebra… and then functions, and how they were completely disregarded because of how unnatural they are. Then, we will see how many years of playing around with polynomials and looking for solutions led to an interesting realization: if we represent these imaginary things by some symbol, we can keep doing algebra and everything works out nicely. A few people decided to embrace the nonsense and they invented some imaginary geometry, and thus were born complex numbers! It turns out that really ugly looking algebraic representations of real systems (e.g. electrical circuits) are easily solvable by passing into the imaginary realm, doing calculations there, and then using only the real results. In this course we will play around with complex numbers. We will do some algebra with complex numbers, and see how they are related to vectors, to circles and triangles, to the number *e*, and to the sine and cosine functions. Some experience with polynomials will be helpful, but not necessary. It may sound really difficult, but it's just complex. ☺ *1-2 stars*

Heavenly Mathematics, Glen (Glen Van Brummelen)

How were the ancient astronomers able to find their way around the heavens without anything even as sophisticated as a telescope? With some clever observations and a little math, it's amazing how much you can infer. Following the footsteps of the ancient Greeks, we will eventually determine the distance from the Earth to the Moon...using only our brains and a meter stick. Along the way, we will develop the fundamentals of the subject the Greeks invented for this purpose, now a part of the school mathematics curriculum: trigonometry. Scientific calculators are encouraged. *2 stars*

The Other Triangular Numbers, Dr. V (Sam Vandervelde)

You are no doubt familiar with the triangular numbers 1,3,6,10,…, which can be thought of as the number of dots needed to fill out an equilateral triangle. In this breakout we will stumble upon a very different, more mysterious sequence of numbers that arise in a similar manner, by looking at how many points in a hexagonal lattice one can enclose with an equilateral triangle. The whole story involves geometry, algebra, complex numbers, and the primes. In particular, it is necessary to have some familiarity with complex numbers and the complex plane in order to participate fully in this breakout. A modest background in number theory, including working with congruences, is also essential. *4 stars*

Week 2, Morning

Eulerian Graphs, Dr. Jane (Jane Butterfield)

What is the best route to take through my neighbourhood this Halloween? Can I walk through this building, locking each door behind me, without getting stuck? How can I search this game's level without missing any treasure chests? Can I embroider this pattern so that it looks good on both sides of the fabric? In this breakout session, we will learn how to model each of these questions with graph theory. Then we will learn -- and prove -- a powerful theorem credited to Euler, as well as some extensions that we will need in order to answer all of these (and more) questions! *Prerequisite: some previous experience with graph theory, such as from taking the Elementary Graph Theory Breakout. 1 star*

Knot Theory, Prof Cahn (Patricia Cahn)

Knots have fascinated farmers, sailors, artists, and myth-makers since prehistoric times, both for their practical usefulness and for their aesthetic symbolism. For over 200 years, mathematicians have intensely studied knots, both for their own intrinsic mathematical interest and for their potential application to chemistry, physics and biology. Today, Knot Theory is an active field of mathematical research with many important applications. It is visual, computational and hands-on, and there are many easily stated open problems. This class will give an introduction to the fundamental questions of Knot Theory. It will take students from simple activities with strings to open problems in the field. It will answer such questions as, “what is the difference between a knot in the mathematical sense and a knot in the everyday sense?,” “how do I tell two knots apart?,” “how can I tell whether a knot can be untangled?,” and “how many different knots are there?” *2 stars*

Hyperbolic Geometry, Dr. Kent (Deborah Kent)

This course explores connections between mathematics and the nature of truth. We'll start with foundational geometrical results that can be developed from a list of four basic assumptions. Adding a familiar definition of parallel lines leads to comfortable facts about Euclidean triangles. But changing what it means for lines to be parallel results in the wildly different reality of non-Euclidean geometry. We'll explore some of these mind-bending results using several models for hyperbolic geometry. *Pre-requisites: Trigonometry. 3 stars*

Number Theory II, Asia (Asia Matthews)

Suppose a fruit stand owner wants to arrange oranges neatly. If the oranges are arranged in rows of 5 then there are 2 left over, if arranged in rows of 6 then there is 1 left over, and if arranged in rows of 7 there are 3 left over. How many oranges could the owner have? Is there a unique solution? Answering this question requires a deeper understanding of modular arithmetic than we achieve in NT1. So in NT2 we discuss inverses mod *m*, the Chinese Remainder Theorem, Fermat's Little Theorem, Wilson's Theorem, the Euler phi-function and Euler's extension of Fermat's Little Theorem. These results will allow us to answer other questions as well, such as: How many ways can 1 be factored? What happens when one repeatedly multiplies a number by itself? And because the answers to such questions are so interesting, we can all enjoy quick divisibility tricks. *2 stars*

Catalan Numbers, Professor Pudwell (Lara Pudwell)

How many ways can you arrange 3 pairs of parentheses legally? The answer (5) can be found by careful listing: ((())), ()(()), (())(), (()()), ()()(). How many ways can you divide a pentagon into triangles by connecting pairs of non-neighboring vertices with line segments? The answer is 5 again! The fact that we got the same answer here is not an accident. The Catalan numbers are the answer to "how many ways can you arrange n pairs of parentheses legally?", "how many ways can you divide a regular (n+2)-sided polygon into triangles?", and literally hundreds of other interesting counting problems. We'll look at the Catalan numbers from several points of view. We'll learn multiple ways to compute the nth Catalan number and see a variety of surprisingly different places where they appear in mathematics. *For this breakout, some previous experience with basic combinatorics will be helpful.3 stars*

Cryptology, Prof Rogness (Jon Rogness)

Have you ever made up a code to send a secret code to a friend, or solved the cryptogram in the newspaper? It turns out there's an entire area of math dedicated to this: cryptology is the study of making and breaking codes. We'll use modular arithmetic to cover classical codes, including shift ciphers, Vigenere ciphers, and substitution ciphers in general. By the end of the week, we'll also cover modern cryptosystems, which are used to keep our information secure online.
Prerequisite: Number Theory I. 2 stars

Week 2, Afternoon

Graph Colouring, Dr. Jane (Jane Butterfield)

Graphs can be coloured by assigning colours to the edge set, or to the vertex set, or even sometimes to both. What makes a colouring *proper*, and how many colours do you need? In this breakout, we will explore some of those questions. We will focus particularly on a class of graphs called *planar graphs*, and will analyze several attempts by mathematicians through the ages to prove the famous Four Colour Theorem. *Pre-requisites: some previous experience with graph theory, such as from taking the Elementary Graph Theory Breakout. Some of the proofs we will explore will use induction, so it would help to be familiar with induction. 3 stars.*

Cardinals and Ordinals, Dr. Hill (Aaron Hill)

This course will deal with comparing sizes in the setting of sets (cardinality) and the setting of well-orders (ordinality). We will prove basic results about the comparability of well-orders without the axiom of choice and similar results about the comparability of sets using the axiom of choice. We will talk about ordinal and cardinal arithmetic and, if time permits, the connections between them in the ZFC axiom system. *Pre-requisites: Some familiarity with the material from the course To Infinity and Beyond would be helpful. 4 stars*

Visual Group Theory, Asia (Asia Matthews)

A GROUP describes symmetries in objects. The Higgs Boson was predicted to exist before it was observed because the known particles interacted as a group, except that a part of the group seemed to be missing. This is very exciting that physicists can use this information to answer questions in the natural world. But from the perspective of pure mathematics, who cares about the applications? The internal structures themselves are so beautiful that we are interested in them in their own right. Look forward to learning visually pleasing representations that help us to classify different finite groups. *2 stars*

Proof by Story, Professor Pudwell (Lara Pudwell)

Armed with a bit of practice and a proper understanding of what certain well-known numbers mean, it becomes possible to prove intimidating identities involving these numbers simply by telling the right story! For instance, in this breakout we will discover that Fibonacci numbers have a combinatorial meaning: they count the number of ways to accomplish a certain task. This view of a Fibonacci number, which goes well beyond the basic algorithm for obtaining a Fibonacci number as the sum of the two previous, enables us to establish a surprisingly large number of clever relationships among them. *2-3 stars*

Spherical Trigonometry, Glen (Glen Van Brummelen)

To find your way through the heavens, along the earth, or across the oceans, you need mathematics. But the math you learn in school mostly takes place on a flat surface --- not the sphere of the heavens, or the earth. To navigate properly, we develop a completely new trigonometry that allows us to find our way around a sphere. We shall discover surprising symmetries and a rich world of theorems, many of which are beautiful and unexpected extensions of some of the most familiar geometric theorems we have learned in school. Scientific calculators are encouraged. *3 stars*

Intermediate Contest Problem Solving, April (April Verser)

In this course, we will be discussing strategies and problem-solving techniques for the AMC 10 & 12. We will focus on problem-solving, while also discussing tricks and strategies for approaching the AMC contests. Topics that will be covered include: AMC strategy, number theory, algebra (including polynomials and functions), geometry, trigonometry, counting and probability, etc. We will also solve challenge problems and work through previous contest problems for practice. *2-3 stars*

Week 3, Morning

Inversions, Prof B (Owen Byer)

An inversion of the plane is a transformation of the points in the plane that preserves various properties of lines, circles, and the way they intersect each other. Inversions are a tool that can be used to provide elegant proofs of many geometric facts. They can be used to create interesting and artistic distortions of figures. Furthermore, properties of inversions can be combined with software like Geogreba to create dynamic figures (for example, Steiner's Porism), and we will explore these interactions. *Students taking this course should have some experience in proving theorems from geometry, though no specific knowledge is necessary. Geogebra experience would be helpful, though it is not required. 3 stars*

Math and Music, Silas (Silas Johnson)

Why do so many people who love math also love music? Perhaps not surprisingly, the two are connected on a deep level in many different ways, and you can "hear" each in the other if you know what to listen for. In this breakout, we'll learn a little bit of math, a little bit of music, and a lot about how they relate to each other. Mathematical topics may include number theory, abstract algebra, trigonometry, combinatorics, and more. On the musical side, we'll try to answer questions like: What is a note, and when are two notes the same? What makes certain combinations of notes sound good together? Why is the 12-tone scale nearly universal in Western music, and what problems is it designed to solve? *1 star*

Combinatorial Games, Kathryn (Kathryn Nyman)

A competitive game or a cooperative (or solitaire) puzzle can provide a fun and engaging pastime. When players interact with a game or puzzle, the players' actions have consequences and can determine whether or not the game is won or the puzzle is solved. One can often use a basic understanding of mathematics to think carefully about good -- or even optimal -- strategies to employ. Turning this around a bit, one can also think of a game or puzzle as a fun and interesting way to explore certain mathematical topics, with the aim of a deeper understanding of the mathematics involved. As we explore this topic together, we'll play some games, try some puzzles, talk about strategies, and have fun learning mathematics together. Relevant mathematical topics include elements of combinatorics, number theory, group theory, and graph theory. *2 stars*

Generating Functions and Sequences, Professor Pudwell (Lara Pudwell)

A sequence is a list of numbers, and the sequences that we are interested in are those with some pattern. For example: consider the sequence 1, 2, 3, 4, 5, 6,... There are many ways to describe this sequence. One (implicit) description is "every number is the previous number plus one". One (explicit) description is "the 100th term is 100". Can you find an implicit or explicit formula for the sequence 1, 5, 14, 30, 55, 91? For some sequences, this question is easy, but usually it's difficult. Sometimes finding an implicit description is easier than finding an explicit one (consider, for example, the Fibonacci sequence). While there aren't techniques that always work, in this course, we will learn a technique that finds an implicit formula for some very nice cases of sequences. This technique involves infinite polynomials (known as generating functions). We'll do some analysis (maybe some long division, perhaps partial fractions) and learn a lot more about these implicit formulas. This course is heavy on algebra, so come prepared to play around with abstract symbols. *3-4 stars*

Discrete and Computational Geometry, Prof Rogness (Jon Rogness)

If you build an art gallery, how many people do you need to guard it? This is a geometric problem, but it has a very different flavor than the geometry questions you've seen in other geometry courses, and we need different tools to solve it — protractors, rulers, or area formulas won't be much help here; the solution turns out to depend on a counting argument based on the number of walls, and not the specific shape of the gallery! During this breakout we'll learn more about the art gallery problem, and also explore related problems which fall under the umbrella of "discrete" and "computational" geometry. We'll learn new ways to compute distances between points — and why we'd care! Once we have those new notions of distance, we'll study the "shortest" way to connect a set of points in the pane. The "computational" in the title means that we'll do some computations based on the geometric objects we're working with; it doesn't mean we'll be using computers, although many of the arguments we'll look at can be implemented with computer programs. In one case, we'll use soapy water to do our computations for us! *Prerequisites. Familiarity with basic Euclidean geometry: distance formula, polygons, and conic sections. Some of our solutions will use proof by induction, so it would help to be familiar with induction. 2 stars*

Marden's Marvelous Theorem, Dr. V (Sam Vandervelde)

You may have heard that there is such a thing as "the cubic formula," and perhaps even looked up the expression online and come to the conclusion that it is considerably uglier than the quadratic formula. This is a travesty, because there is actually an elegant way to present Cardano's cubic formula, involving complex cube roots of unity. We will begin by understanding this method, then move on to tackle Marden's Theorem, which states that if one plots the three roots of an arbitrary cubic polynomial in the complex plane, draws the triangle having these three points as vertices, then inscribes an ellipse tangent to the sides of this triangle at their midpoints, then the roots of the derivative of the cubic will be situated precisely at the foci of this ellipse. Our main tool for unraveling this mystery is linear transformations of the complex plane, so it is essential to have a working knowledge of complex numbers. However, no background in calculus is needed or assumed. *4 stars*

Week 3, Afternoon

The Binomial and Poisson Distributions, Prof B (Owen Byer)

The Poisson Distribution is used to find the probabilities of the number of occurrences of some event in a fixed interval of space or time. Typical questions include the following.

1. What is the probability that there are two typographical errors on a random page of a given novel?
2. What is the probability that three different calls are made to an Emergency Response Unit in a given hour?

We will derive the formula for the Poisson probability distribution from beginning principles, discovering the binomial distribution and the number *e* along the way. We will examine many applications to real world phenomena. *Students taking this course should have experience with basic probability and familiarity with combinations and permutations will be very helpful. 2-3 stars*

Game Theory, Dr. Kent (Deborah Kent)

This class will start with a game of Hex and end with a proof of Nash's Equilibrium Theorem, one of the triumphs of classical game theory. In between, we'll explore a nice blend of of continuous and combinatorial mathematics, including the elegant and useful Sperner's Lemma. Class will also incorporate a dash of topology with some definitions and an introduction to Brouwer's Fixed Point Theorem. We'll be able to draw and visualize deep mathematical ideas in a 2D-context and discuss generalizations to higher dimensions. If there's time, we can even put it all together to determine a fair way to divide a cake! *4 Stars*

Compositions, Partitions and Fibonacci Numbers, Kathryn (Kathryn Nyman)

How many ways can one you write 4 as a sum of positive integers? If order matters, so that we consider 3+1 and 1+3 to be different, then the answer (8) is an easy application of basic combinatorics. But if we only care about the (unordered) *set* of addends, the answer is 5. This simple question leads to a rich, elegant and surprisingly challenging mathematical area on the intersection of combinatorics and number theory. We will look at integer partitions from several vantage points and take detours to visit the Fibonacci numbers. Along the way we will encounter beautiful bijections, clever counting, and tricky tilings. For this breakout, some previous experience with elementary combinatorics will be helpful. *3 stars*

Counting Permutations, Professor Pudwell (Lara Pudwell)

A permutation is a list of the numbers 1, 2,..., n where the order matters. There are 6 permutations of 1, 2, 3 (can you list them?), and in general there are n!=n*(n-1)*(n-2)....*2*1 permutations of 1, 2,...,n. That's the warmup problem. In this class we'll look at helpful ways to visualize permutations (other than writing them as lists) and consider questions like "how many permutations of 1, 2, ..., n have k descents?" and "what does it mean for a larger permutation to have a smaller permutation inside of it?". Familiarity with basic combinatorics (permutations, combinations, and recurrences) will be helpful. *4 stars*

Rigidity Theory, Maddy (Maddy Ritter)

This breakout will use interactive materials, Cinderella software, linear algebra, and graph theory to look at rigid frameworks in 2-space. We will explore motions, Henneberg moves, and how to express graphs in matrix form to further our understanding of them. *Prerequisites: Students should be familiar with matrices and basic graph theory. 3 stars*

The Shape of Space, Prof Rogness (Jon Rogness)

This breakout will explore ideas in the area of math known as topology, where a donut (torus) and coffee cup are equivalent objects. It will provide a nice followup for any students who have done the previous breakouts this year which covered the torus, although those courses are not prerequisites for this one. We'll start by learning a few basic rules in topology. Then we'll look at the torus, either as a review or brief introduction depending on whether students were in those previous breakouts. Then we'll look at how to construct other two dimensional surfaces, like the Mobius strip and Klein Bottle. As it turns out, in one of the triumphs of mathematics, we can describe how to build all of the so-called "closed surfaces" by sewing together spheres, Moebius strips and donuts. After we deal with surfaces, we'll have the necessary skills to move on to three dimensions. That means we can think about different possible shapes for the universe -- i.e. the Shape of Space -- and explain why the picture below with the dodecahedra might be a model of how our universe is put together. There won't be much in the way of computations or algebra in this breakout, but you should like thinking about three dimensional shapes. (For example, if you like looking at nets and figuring out what the resulting shape is, we'll be doing more complicated versions of that process.) *2 stars*

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