Problem 6

You underestimated Problem 6.

We will use it not so much to illustrate writing as to illustrate developing a theory.

First, you were confused what it meant. You and referee given a graph – vertices (dots) and edges.

You pick how many markers you get.

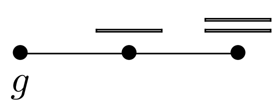
Referee picks where they are put and which vertex is the goal.

You try to move a marker onto the goal, through a sequence of moves one edge at a time, with the rule each time you move a marker from a vertex you must discard another marker *from that vertex* (so there had to be at least 2 at that vertex)

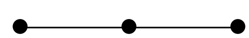
Example; given graph P_3: 

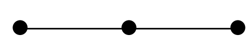
I choose 3 markers.

Referee chooses this arrangement and goal:

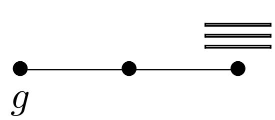


I win moving as follows

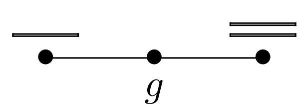




What if the referee had chosen



or



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Notation: For a graph G let

$$\begin{aligned} \mathcal{P}(G) &= \text{the min number of markers for } G \text{ guaranteeing} \\ &\quad \text{a win for any configuration and goal} \\ \mathcal{P}(G, g) &= \text{the min number of markers for graph } G \text{ and} \\ &\quad \text{goal vertex } g \text{ guaranteeing a win for any con-} \\ &\quad \text{figuration and that goal.} \\ \mathcal{P}(G) &= \max_{g \in G} \mathcal{P}(G, g). \end{aligned}$$

Typical answer (student paper)

So claim; for $G = P_n$ the unique case for needing the most markers is to let g be one end vertex and the configuration is to put all the markers at the other end, in which case you need 2^{n-1} markers, which will give you a win if you always move the markers towards the goal.

Well, min maybe not all at the end:

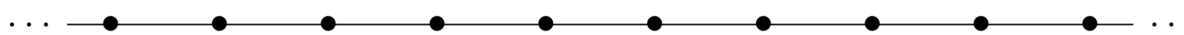
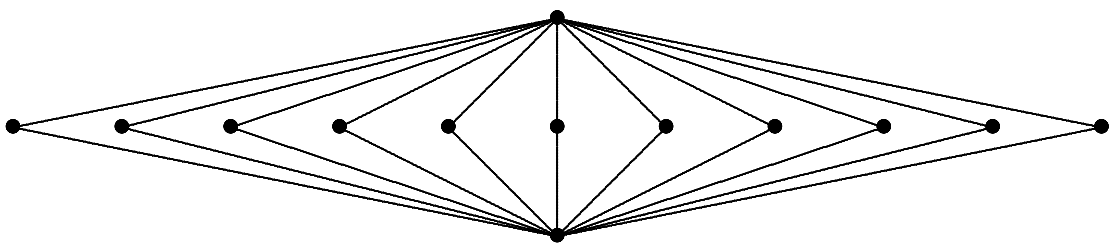


figure where worst min definitely not at end.



This is not a path graph but where was pathgraph-iness used in the argument? Answer: it wasn't, so the student "proved too much".

Yet it is true: $\mathcal{P}(P_n, v_1) = 2^{n-1}$.

One way pathgraph-iness could be used: induction

(sketch; I'll give a different sort of proof in a minute)

Lot of questions have arisen already

- When will less than 2^{n-1} markers win on P_n with $g = v_1$? Is there a *measure* of the winningness of configurations?
- Is there an algorithm which wins whenever the configuration can win?
- what about $\mathcal{P}(P_n)$?

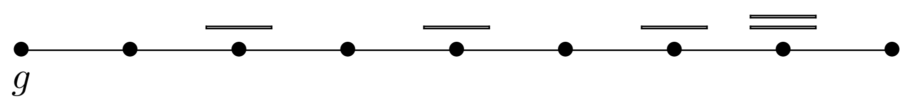
- what about $\mathcal{P}(G)$ for other graphs?

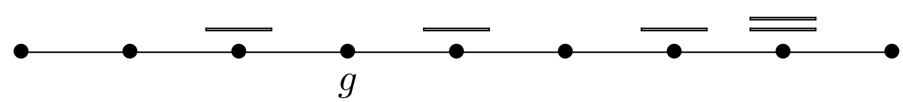
Sophisticated argument for deciding if a configuration can win, Give each marker a **positional value**

The **value of a marker** given a goal is 2^{-d} where d is the distance of the marker from the goal vertex

distance = number of edges on shortest path (the path with the fewest edges)

The **value of a configuration** given a goal = the sum of the values of the markers given that goal.





Lemma With each legitimate move the value of the configuration stays the same or goes down.

Proof:

Theorem (The necessary value theorem) A necessary condition for a configuration to be winnable is that the value be ≥ 1.

Proof: Equivalently show if $v(\mathcal{C}) < 1$ then can't win.

Good news: necessary for any graph (nothing about nature of the graph used in the proof)

Why necessary?

Those of you who introduced this mostly claimed it is also sufficient

Now bad news: not sufficient

Good news: Is sufficient if $G = P_n$ and goal at the end.

Theorem. Let $G = P_n$ and let the goal be an endpoint. Then a sufficient condition for winnability is that the value of the configuration be ≥ 1.

Proof: By induction, tricky when odd number of markers at other end, but show the value is at least $1/2^{n-1}$ greater than 1. ∎

Corollary. Again Let $G = P_n$ and let the goal be an endpoint. An algorithm for winning each winnable configuration is to start at the opposite end and move as many markers one vertex forward as you can, then repeat one step closer to the goal.

Good news: even more general "semi-algorithm" works for P_n with goal an endpoint. wherever find ≥ 2 markers on a vertex, discard one, move other towards the goal. Continue until you have either won or are stuck (no vertex with 2 markers)

Theorem. If a configuration is winnable on P_n with goal at end, the above semi-algorithm will win it

Proof: Procedure just described can't continue indefinitely because can't repeat a configuration (why?) and only a finite number of configurations with at most the number of markers you started with and with at least one vertex with two markers. So eventually it wins or it gets stuck with at most one marker on each vertex. Since we started with a winnable configuration we know by a previous thm that $v \geq 1$. We also know the value of the final configuration equals the original value since we always moved toward the goal. But if there is at most one marker on each vertex (and none on the goal) the value is at most

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = 1 - \frac{1}{2^n} < 1.$$

Contradiction, so the algorithm must terminate the other way, with a win. ∎

We can use this knowledge of P_n with goal at one end , that is, $\mathcal{P}(P_n, v_0)$, to determine $\mathcal{P}(P_n)$.

Ask class. *Hint*: there are only two sides of the goal and you can't combine efforts on both to win.

We can also use this knowledge of P_n with goal at one end to begin to understand other graphs. Example. Even though $v \geq 1$ is not necessary and sufficient for winnability for P_n with goal not at the end, the algorithm just described still works if and only if the game is winnable.

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Various claims you made are false for the game described but correct for the simplified game where you can move fractional markers but each time you move something you throw half of it away. Call this the **lose half nonintegral** version

Theorem. Fr any G and g, the lose half nonintegral version is winnable iff $v \geq 1$.

Proof: At each vertex in turn, move forward half of what's there towards the goal by shortest route, discarding the rest. Value of configuration remains the same in the end, all parts of pieces are on the goal, and the value therefore equal the (fractional) number of pieces on the goal. ∎

Corollary. For any graph min # to ensure winnability of the lose half nonintegral version is 2^d, where d is max distance to the goal, and the only configurations which require 2^d markers are configurations which put all the markers on vertices distance d from the goal. If there in only one such vertex the worst configuration is unique. ∎

So most of you had the right answer, but to a different game!

Makes me think: what other simplifications should we consider?

Integral (1) and nonintegral (2) versions of:

a. Move half of what you have on a vertex, throw away half.

b. move half of what you have on a vertex, keep the other half.

c. move markers from a vertex, so long as you have as many markers somewhere to throw away

d. move markers from a vertex, so long as you have as many markers somewhere.

Original problem is 1a. We just considered 2a. I say consider 1b next. Why?

Theorem. For game 1b, for any connected graph, $\mathcal{P}(G) = |V|$, the number of vertices.

Proof sketch: $V - 1$ markers has a non winnable configuration. With V markers, either you already won or, by pigeon hole principle, at least one double. Move the top marker closer to the goal. Repeat. This procedure will terminate in win.

Why?

- doesn't go on forever – why?
- either ends in win or ends in a most one marker per vertex
- but can't end in latter

Back to the original problem with both conditions on a move. The question is which wins out when.

Theorem $\mathcal{P}(C_{2n}) = 2^n$

but

Theorem $\mathcal{P}(C_{2n+1}) \neq 2^n$

In fact, $\mathcal{P}(C_{2n+1}) \sim \frac{4}{3} 2^n$.