Problem 7b, General Case

Let N be the initial number in the game. For instance, let

$$N = 2^1\,3^2\,5^3\,7^4\,11^5$$

Let m_0 = number of distinct primes in the prime factorization of N that are raised to a power = 0 mod 3.

Above $m_0 = 1$, because 5 is the only prime raised to a multiple of 3.

Let m_1 = number of distinct primes in the prime factorization of N tht are raised to a power = 1 mod 3.

Above $m_1 = 2$. The primes are 2 and 7. The powers are 1 and 4.

Similarly, $m_2 = 2$. The primes are 3 and 11.

Now consider

$$N' = 2^1\,3^2\,5^3\,7^4\,11^6$$

$$m_0 =$$

$$m_1 =$$

$$m_2 =$$

As many of you noted, the actual primes make no difference. What matters is how many powers there are of each one. So the game can be restated as follows: there are several piles of stones, with α_i stones in pile i. On each turn, a player can remove 1 or 2 stones from any one pile. (A player can only remove 2 if there are at least 2 in the pile.) The player who is supposed to move when there are no stones left loses.

What's the definition of m_j with the stone language?

June 22, 2018

By the **3-modularity** of a pile of stones (or any finite set of objects) we mean the value mod 3 of the number of stones.

Similarly there is k-modularity. What is a more common word for 2-modularity?

Restate the definition of m_j using the word 3-modularity.

Theorem 2. In the game of problem 7b, Suppose a player has to divide the number N. If both players play optimally, that player wins if and only if at least one of m_1 and m_2 is odd.

Example. According to the theorem, N above is a loss for the first player, because m_1 and m_2 are both even.

However, N' above is a win for the first player, because, $m_2 = 1$ (What's a winning first move for the first player?)

Proof: Let $\mathcal{W}$ be the set if numbers N which are claimed to be winning in the theorem statement. Let $\mathcal{L}$ be all other positive integers, ie., the numbers for which both m_1 an m_2 are even. We must show.

1) The game ending position, $N = 1$, is in $\mathcal{L}$
2) From $\mathcal{L}$, every legitimate move leads to $\mathcal{W}$.
3) For every $N \in \mathcal{W}$ there is some move which results in a number in $\mathcal{L}$

Henceforth think in terms of stones.

On each turn whatever pile of stones the player removes 1 or 2 from, the 3-modularity of that pile will change. Let j be the 3-modularity of that pile before the player moves and let k be the 3-mdoulariy afterwards. Then the effect of the move it to decrease m_j by 1 and increase m_k by 1. In particular, the move changes the parity of exactly two of m_0, m_1, m_2. We don't care about the parity of m_0, so each moves affects the parity of one or two of the values we care about.

This already proves 2). Why?

It's easy to see that $1 \in \mathcal{L}$. Why?

As for 3), say only m_1 is odd. Because 0 is not odd, there is at least one pile with 3-modularity $= 1$. Remove one stone from that pile; m_1 is now even. The parity of m_0 has also changed, but we don't care about that.

If m_2 alone is odd ....

If both m_1, m_2 are odd, there exists at least one pile with 3-modularity 2. Remove one stone from it, giving it 3-modularity 1. Now m_2 has gone down by one, m_1 has gone up by one; hence they are both even. ∎

Warmup. Show that Theorem 1 is a special case of Theorem 2.

Challenge. Solve 7c: Same game, but in each move you can remove up to s stones from any one pile. (In 7a, $s = 1$; in 7b, $s = 2$.)