Plenary Outline

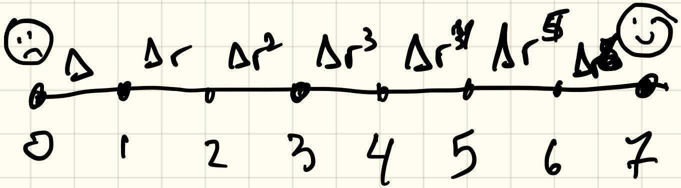
180716

I. Intro: Probability, Expectation, Random Walks

- Gambler's Ruin: Don't be a sucker
- Sucker Bets
- Expectation — * cycles
- Recursive Prob
- Lotteries
- St Pete
- Polya's Theorem about Prob of Return

The classic RW

180716



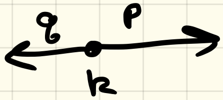
$P_k := P(k \to 7$ without $0)$

$0 \leq P(A) \leq 1$

$P(A \text{ and } B) = P(A) \cdot P(B|A)$

$a_k = p a_{k+1} + q a_{k-1}$

$p a_k + q a_k = p a_{k+1} + q a_{k-1}$



$P_0 = 0$

$P_1 = 1$

$P_k =$

$P_k = p P_{k+1} + q P_{k-1}$

$q \Delta_k = p \Delta_{k+1}$

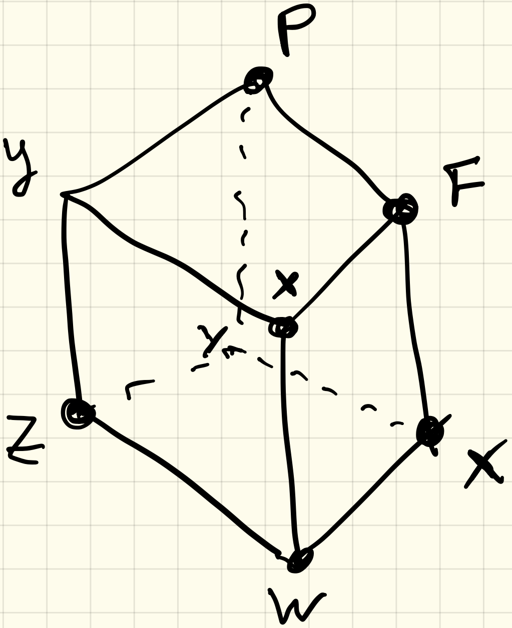
$\Delta_{k+1} = \frac{q}{p} \Delta_k$

$P_k - P_{k-1} = p P_{k+1} + (q-1) P_{k-1}$

$\Delta \left(\frac{r^6 - 1}{r-1} \right) =$

$\Delta \frac{(r^{2N} - 1)}{r-1} = P = 1$

$P_N = \Delta \frac{r^N - 1}{r-1} = \frac{1}{r^N + 1}$



$$x = \frac{1}{3} + \frac{1}{3}y + \frac{1}{3}w$$

$$y = \frac{1}{3}x + \frac{1}{3}z$$

$$w = \frac{2}{3}x + \frac{1}{3}z$$

$$z = \frac{2}{3}y + \frac{1}{3}w$$

$$3x = 1 + y + w$$

$$3x - y - w = 1$$

$$-x + 3y - z = 0$$

$$-2x - z + 3w = 0$$

$$-2y + 3z - w = 0$$

$$2y - 2z + 2w = 1$$

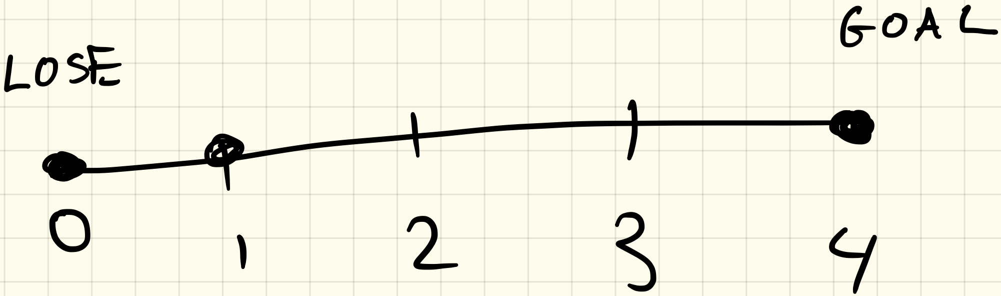
$$z + w = 1$$

СДЕЛУЕМ АМЕРИКУ СНОВА ВЕЛИКОЙ

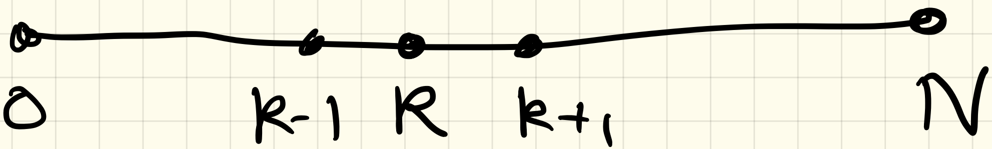
Sucker Bet

I'll bet u $5 That if u give me $10, I'll give u $20

Random Walk with Boundary



$\frac{1}{4}$ $\frac{1}{2}$ $\frac{3}{4}$



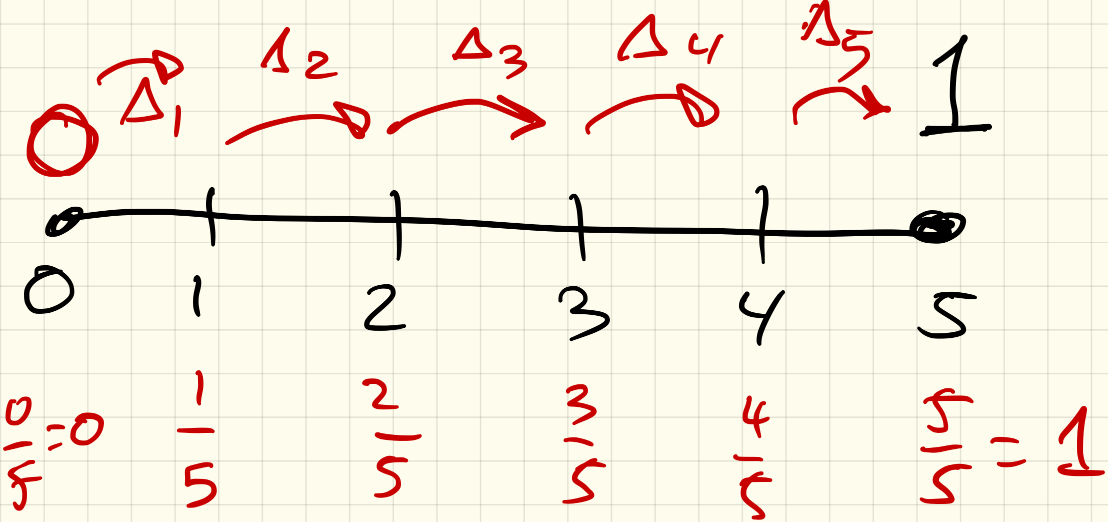
$P_k := \text{PROB}(\text{"WIN"}, \text{starting @ } k)$

$P_0 = 0 \qquad P_N = 1 \qquad \Delta_k := P_k - P_{k-1}$

$$P_k = \frac{1}{2} P_{k+1} + \frac{1}{2} P_{k-1}$$

$$2P_k = P_{k+1} + P_{k-1}$$

$$P_k - P_{k-1} = P_{k+1} - P_k$$



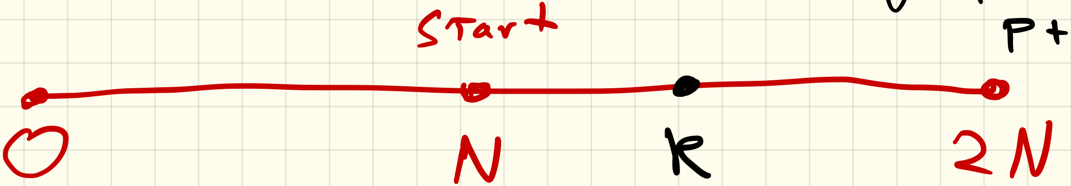
0
Δ1
Δ2
Δ3
Δ4
Δ5
1
0
1
2
3
4
5
0/5 = 0
1/5
2/5
3/5
4/5
5/5 = 1

Start with N

p = prob of win a bet

$q = P(\text{lose bet})$

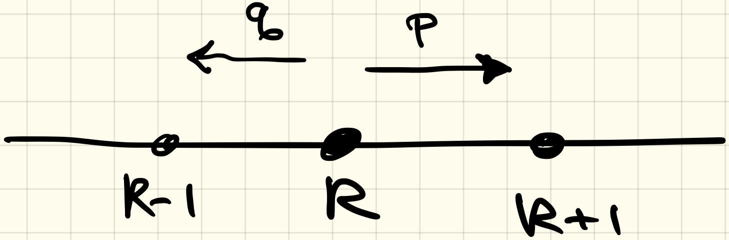
$p + q = 1$



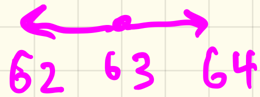
$a_k = P(\text{win, start @ } k)$

$a_0 = 0$

$a_{2N} = 1$



$$a_k = p \cdot a_{k+1} + q\, a_{k-1}$$



$N = 100$ $k = 63$

$a_{63} = .51 a_{62} + .49 \cdot a_{64}$

$$r := \frac{q}{p}$$

$$(p+q)a_k = p \cdot a_{k+1} + q a_{k-1}$$

$$p a_k + q a_k = p a_{k+1} + q a_{k-1}$$

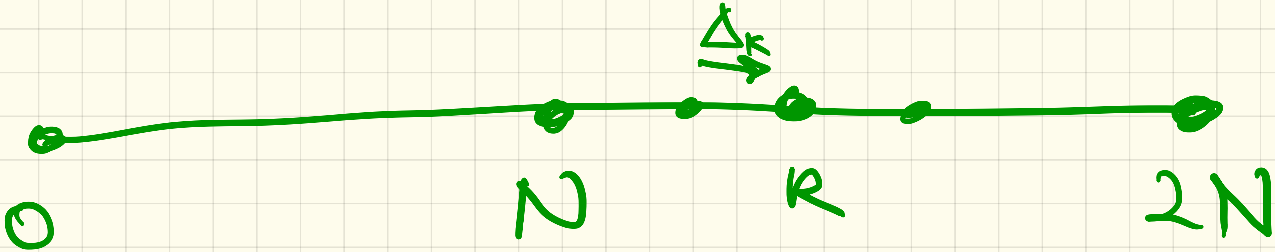
$$q(a_k - a_{k-1}) = p(a_{k+1} - a_k)$$

$$q \Delta_k = p \Delta_{k+1}$$

$$\Delta_{k+1} = \frac{q}{p} \Delta_k$$

$$\Delta_{k+1} = r\,\Delta_k \qquad a_N$$

$$r = \frac{q}{p}$$



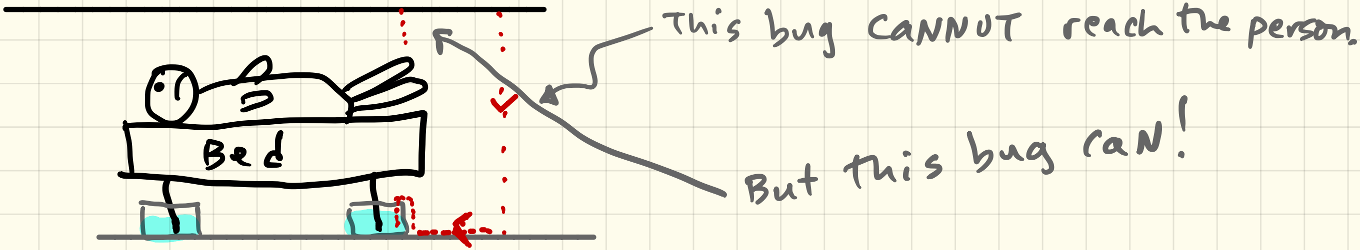
$$a + ar + ar^2 + \dots + ar^{m-1} =$$

POD

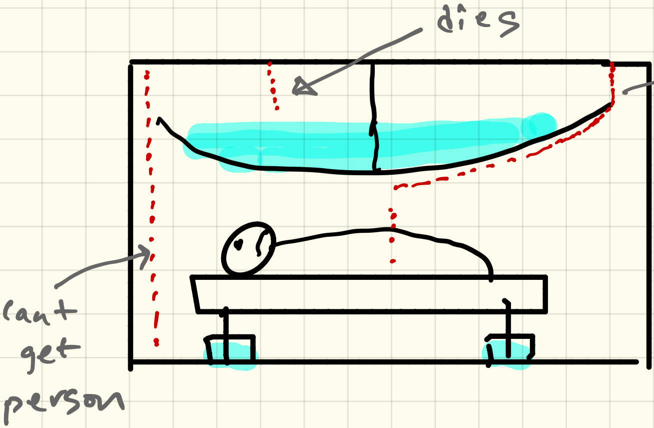
math bug

July 17, 2018 "Bug Problem"

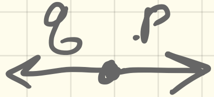
The bugs have zero diameter – they are points – but they sting if they land on you. They randomly drop vertically from the ceiling and then can crawl on any surface. However, water kills them. Your goal: create a barrier that keeps ALL bugs away. You are also allowed to fill 4 coffee cans with water to keep bugs from crawling up like this:

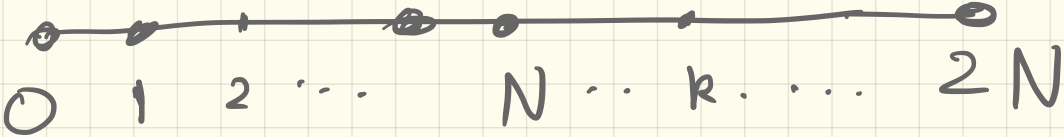


Your barrier cannot be air tight: The person must get air from the ceiling. Water will stop air (person is not a fish). Obviously your barrier MUST be bigger than the bed. Here is a try that fails:



This bug hits the exact edge of the barrier, crawls along the bottom, and drops on the person!!





$a_0 = 0$

$\Delta_k := a_k - a_{k-1}$

$a_{2N} = 1$

$\Delta_{k+1} = \Delta_k r$

$r = \frac{q}{p}$

geo prog

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

$$S - rS = a - ar^n$$

$$S(1-r) = a - ar^n$$

$$S = \frac{ar^n - a}{r - 1}$$

$$\Delta = \Delta_1$$

$$\Delta + r\Delta + r^2\Delta + \cdots + r^{2N-1}\Delta = 1$$

$$\Delta\left(1 + r + \cdots + r^{2N-1}\right) = 1$$

$$\Delta\left(\frac{r^{2N} - 1}{r - 1}\right) = 1 \Rightarrow \Delta = \frac{r-1}{r^{2N}-1}$$

$$a_N = \Delta + r\Delta + r^2\Delta + \cdots + r^{N-1}\Delta$$

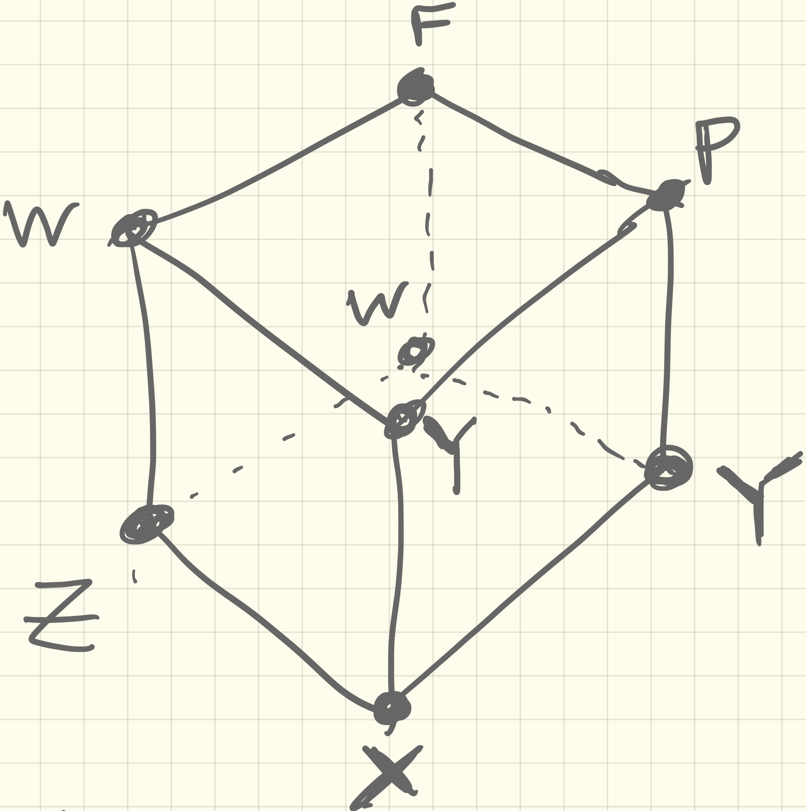
$$= \Delta\left(1 + r + r^2 + \cdots + r^{N-1}\right) = \Delta\left(\frac{r^N - 1}{r - 1}\right)$$

$$\Delta = \frac{r-1}{r^{2N}-1}$$

$$a_N = \Delta\left(\frac{r^N-1}{r-1}\right)$$

$$a_N = \frac{1}{r^N+1}$$

$$a_N = \frac{r^N-1}{r^{2N}-1} = \frac{r^N-1}{(r^N-1)(r^N+1)}$$



$$P = 0$$
$$F = 1$$
$$Y = \frac{1}{3} X + \frac{1}{3} W$$
$$X = \frac{2}{3} Y + \frac{1}{3} Z$$
$$Z = \frac{2}{3} W + \frac{1}{3} X$$
$$W = \frac{1}{3} + \frac{1}{3} Y + \frac{1}{3} Z$$

$\frac{9}{14}$ $\frac{5}{14}$ $\frac{3}{7}$ $\frac{4}{7}$

Penny's Game

1969

Martin Gardner

Collassal (sp?) Book of Math

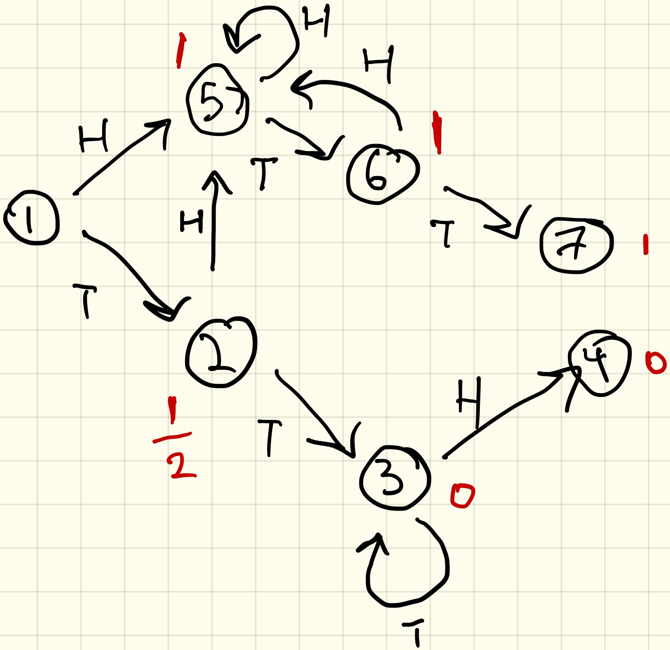
TTT — victim

HTT — me ☺

Victim: TTH

Dr. Evil: HTT

HTT vs TTH



$P_3 = \frac{1}{2} P_3 \Rightarrow P_3 = 0$

$P_2 = \frac{1}{2} P_5$

$P_6 = \frac{1}{2} + \frac{1}{2} P_5$ ⇄ $P_6 = 1$

$P_5 = \frac{1}{2} P_5 + \frac{1}{2} P_6 \Rightarrow P_5 = P_6$

$P_1 = \frac{1}{2} P_2 + \frac{1}{2} P_5$

THT vs HTH

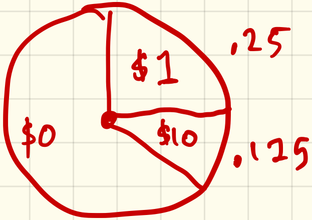
TTH $\frac{2}{3}$ vs THH

7/19/18

① Penny game computation

② Exp. of a RV

E(Lottery)



$$E(X) = \sum_{v \in \text{vals}} v\, P(X=v)$$

③ E(Fixed pt) = 1

④ E("system")

⑤ E(stop time)

all dice:

$$1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + 6$$

(2 3 4 5 6)

⑥ St Pete

If victim says

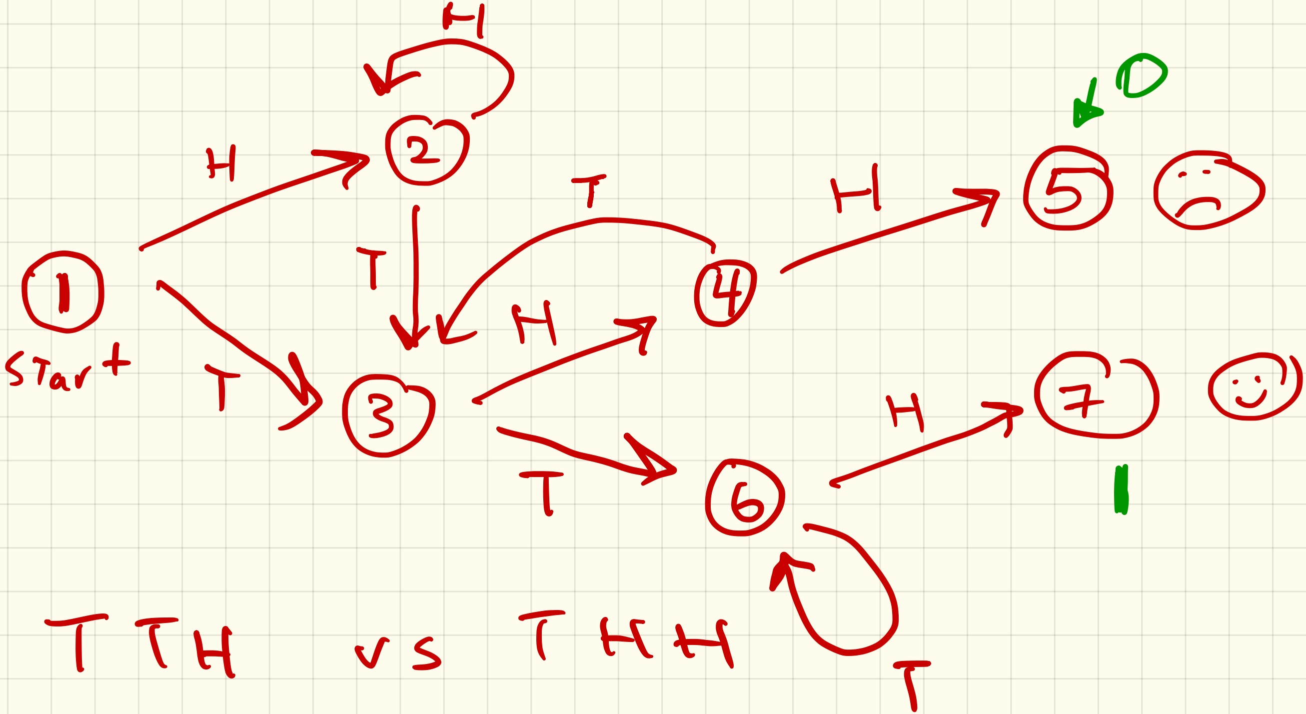
XYZ

Dr Evil replies $\bar{Y}XY$

victim: THH

Evil TTH

$$P(\text{TTH comes first vs THH}) = \frac{2}{3}$$



TTH vs THH

$P_k := \Pr(\text{EVIL wins, starting at } k)$

$P_4 = \frac{1}{2}P_5 + \frac{1}{2}P_3$

Random Variable

$X :=$ # of heads when you toss a fair coin 100 times

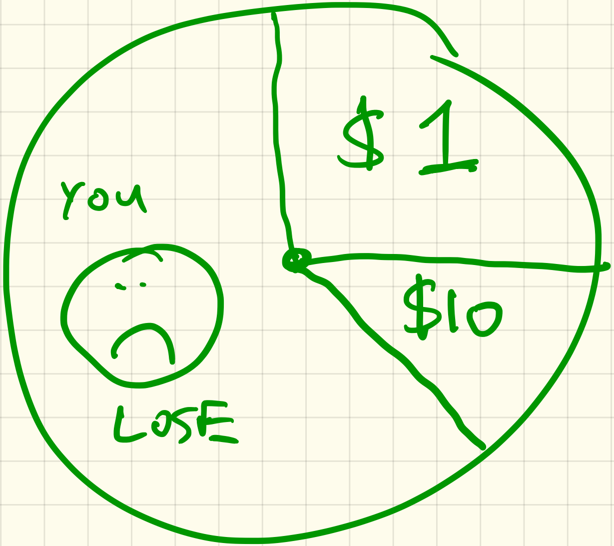
$$P(X = 64) = \binom{100}{64} \frac{1}{2^{100}}$$

Average value of X?

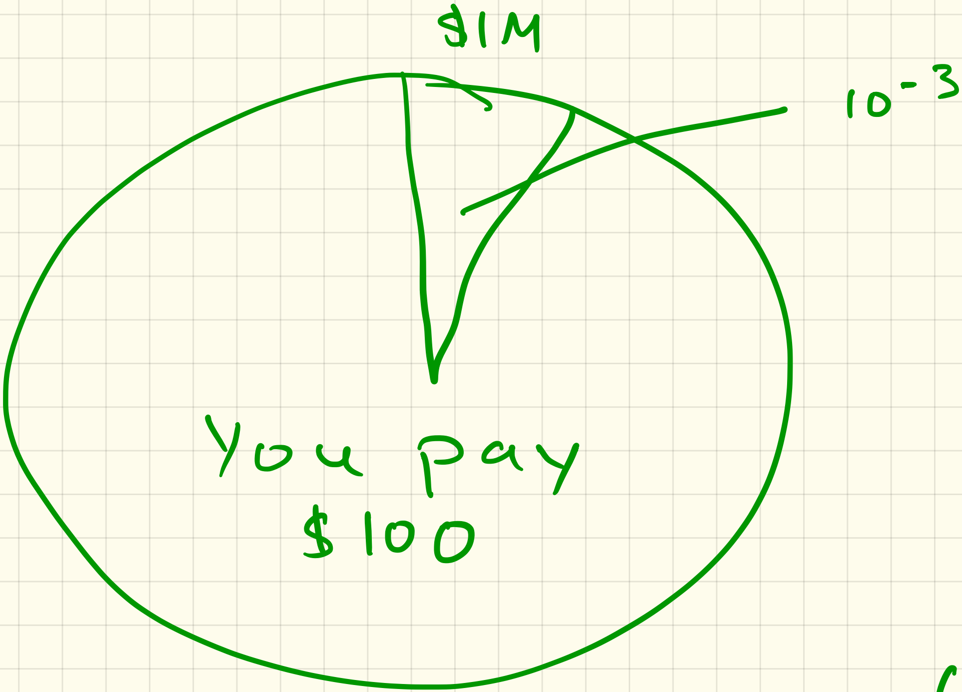
Expected value $\quad E(X) = 50$

L = lottery ticket

$$E(X) = \sum_{v \in \text{outputs}} v \cdot P(X=v)$$



$$1 \times P(L=1) + 10 \times P(L=10) = 1.5$$



$$E = .999 \times (-100) + \underbrace{.001 \times 10^6}_{1000}$$

$$-99.9 + 1000$$

$$E(X+Y) = E(X) + E(Y)$$

$X + Y$

$2 \cdot D =$

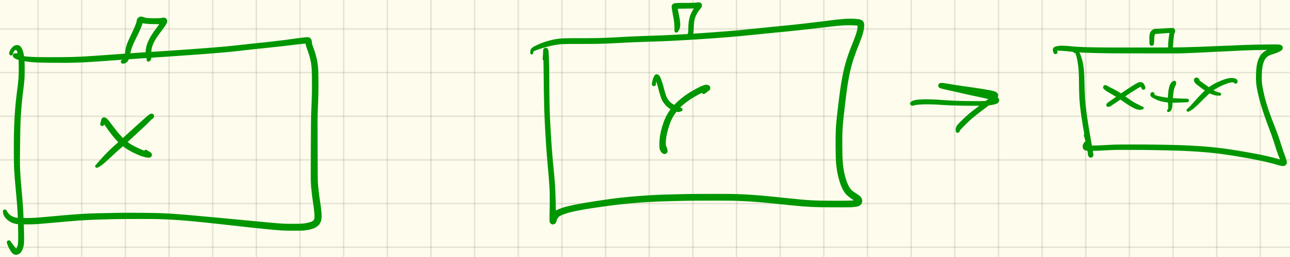
2, 4, 6, 8, 10, 12

$X + X \neq 2X$

D = die

D + D = sum of two dice

2, 3, . . . , 12



L = # of people who get their lanyard

$L = 0, 1, 2, 3, \ldots, 103, 105$

$$L_1 = \begin{cases} 1 & \text{if camper \#1 is ☺} \\ 0 & \text{otherwise} \end{cases}$$

$$E(L_1) = 1 \cdot P(L_1 = 1) + 0 \cdot P(L_1 = 0) = \frac{1}{105}$$

$$L_1 + L_2 + \cdots + L_{105} = L$$

$$E(L) = E(L_1) + E(L_2) + \cdots + E(L_{105})$$

$$= \frac{1}{105} + \frac{1}{105} + \cdots$$

$$= 1$$

Coin game   St. Petersburg Game

TTH → win \$8

H → win 2

TTTTTTTH → win \$128

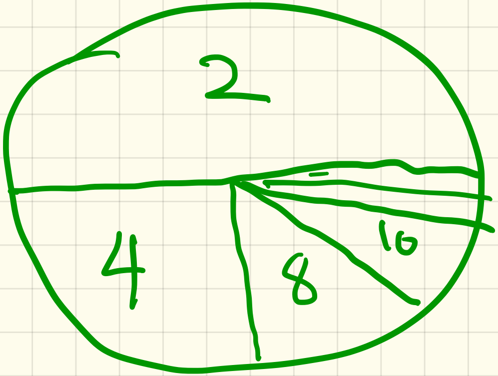
G = 2, 4, 8, 16, ...

SCENARIO	PROB	WIN
H	$\frac{1}{2}$	\$2
TH	$\frac{1}{4}$	\$4
TTH	$\frac{1}{8}$	\$8
⋮		
$\underbrace{T \cdots T}_{k}H$	$\frac{1}{2^{k+1}}$	2^{k+1}

$$E(G) = \infty$$

$$\sum \frac{1}{2^k} \cdot \frac{2}{2^k}$$

$$\sum \frac{1}{2^k} \cdot 2^k = \sum 1$$



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- St. Petersburg Simulation
- ∞ Random Walks

  Polyas Theorem

  Recurrent vs. Transient

- Recurrence $\Leftrightarrow$ $E(\#\text{returns}) = \infty$

  (Recurrence: $P(\text{return}) = 1$)

Let $u = P(0 \circlearrowleft)$. Then $P(\underline{\text{exactly}}\ k \text{ returns}) = u^k(1-u)$

$$\Rightarrow E(\#\text{returns}) = \sum_{1}^{\infty} k\, u^k (1-u) = \frac{1}{1-u}$$

$n = 10$

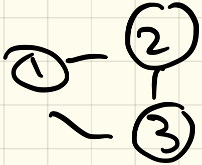
C = # of cycles in a random perm.

$E(C)$

$\pi = (1\,2\,3)(4\,5\,6)(7\,8\,9)(10)$

(4 5 6): 3-cycle / tricycle

(10): unicycle



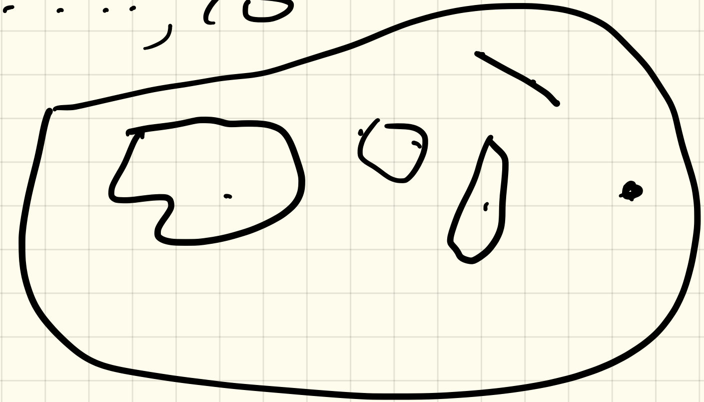
π:

1	2	3	4	5	6	7	8	9	10
↓	↓	↓	↓	↓	↓				↓
2	3	1	5	6	4				10

$\theta = (1)(2)(3) \cdot \cdots (10)$

$n = 10$

$C = 1, 2, \cdots, 10$



$P(C=10) = \frac{1}{10!}$

$P(C=1) = \frac{1}{10}$

$C = \sum \cdots$

$C_k :=$ # of k-cycles

$n=10$ $C_1 + C_2 + C_3 + \cdots + C_{10} = C$

$= \#\text{UN1} + \#\text{bic} + \#\text{tri}, \ldots + \ldots \#_{10\text{-cyc}}$

$E(C_1) = 1$

$U_k := \begin{cases} 1 & \text{if } k \text{ is in a bicyc} \\ 0 & \text{if not} \end{cases}$

$E(C_2) = ??$

$C_2 = U_1 + \cdots + U_{10}$

$n = 10$

$\pi = (1\ x)\overbrace{(\ \cdot\ -\ -\ -\ \cdots\ )}$

$\uparrow$

$\underbrace{2-10}_{9}$ choices

choices $8!$

choices $= 9!$

perms $= 10!$

$\frac{9!}{10!} = \frac{1}{10}$

~~[illegible]~~ $n = 3$

			u_2	c_1	c_2	c_3
1 2 3	→	(1)(2)(3)	0	3	0	0
1 3 2	→	(1)(2 3)	1	1	1	0
2 1 3	→	(1 2)(3)	1	1	1	0
2 3 1	→	(1 2 3)	0	0	0	1
3 1 2	→	(1 3 2)	0	0	0	1
3 2 1	→	(1 3)(2)	0	1	1	0
			2	6	3	2

u_2

$$E(u_2) = \frac{1}{3}$$

$$E(u_1) = \frac{1}{3}$$

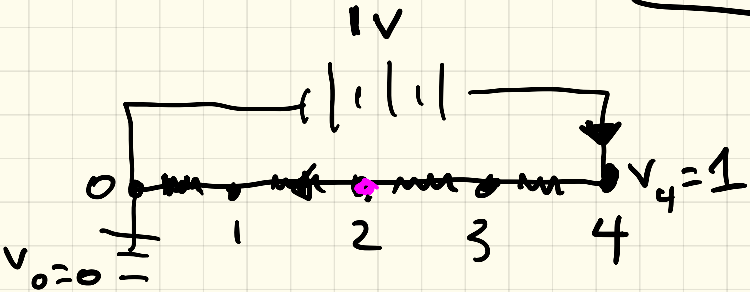
$$E(u_3) = \frac{1}{3}$$

$$E(c) \simeq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

$$\approx \log n$$

$$\log_e n$$

$$V_x = \frac{V_{x+1} + V_{x-1}}{2}$$



Ohms Law

Kirchoff's Law

$$i_{xy} = \frac{v_x - v_y}{R}$$

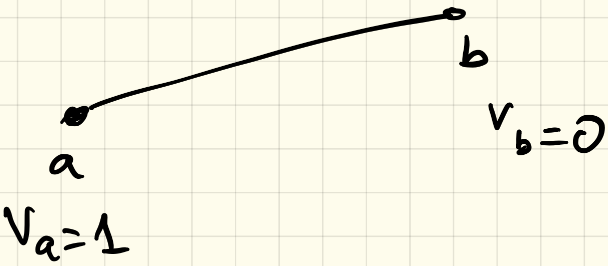
i = current

R = resist

v_x = voltage at x

Elect. Circuit / RW

180707



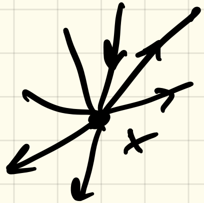
Ohm $\rightarrow$ $i_{xy} = \dfrac{V_x - V_y}{R_{xy}}$

$i_{xy} = (V_x - V_y) C_{xy}$

R_{xy} = resist

$C_{xy} := \dfrac{1}{R_{xy}}$ = conductance

i_{xy} = current



$i_x := \sum_y i_{xy}$

$C_x := \sum_y C_{xy}$

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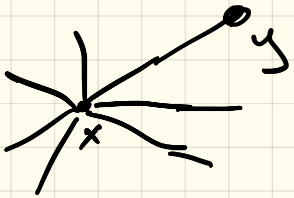


$i_x = 0$

$$P_{xy} := \frac{C_{xy}}{C_x}$$

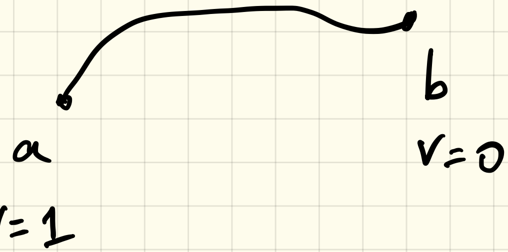
$P(x \to y)$ in a step

$v_x := P(x \longrightarrow a$ without visit $b)$

$$0 = \sum_y i_{xy} = \sum_y (v_x - v_y) C_{xy}$$

$$v_x C_x = \sum_y v_y C_{xy}$$

$$v_x = \sum_y \frac{C_{xy}}{C_x} v_y$$



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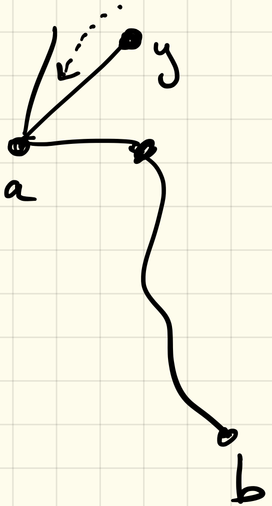
$$i_a = \sum_y i_{ay} = \sum_y (V_a - V_y) C_{ay}$$



$$= C_a \sum_y (V_a - V_y) \frac{C_{ay}}{C_a}$$

$$= C_a \Big(\underset{1}{V_a} \underbrace{\sum_y P_{ay}}_{1} - \sum_y V_y P_{ay} \Big)$$

$$= C_a \Big(1 - \sum_y \underline{V_y P_{ay}} \Big)$$

a to y … eventually to a, avoiding b

$$= C_a \big(P(\text{escape } a \text{ to } b) \big)$$

$$\frac{V_a}{i_a} = R_E \text{ (eff resist)}$$

$$= \frac{1}{i_a} = \frac{1}{P(a \to b \text{ escape})}$$

180721

I. Recap E(cycles of a random perm)

II. ∞ Random walks

III. The Role of physics

$$E(\text{unicycles}) = 1$$

$n = 5$

$$u_1 := \begin{cases} 1 & \text{if person \#1 in unicycle} \\ 0 & \text{otherwise} \end{cases}$$

$$P(u_1 = 1) = \frac{1}{5} \qquad E(u_1) = \frac{1}{5}$$

$$u := u_1 + u_2 + \cdots + u_5$$

$$E(u) = E(u_1) + E(u_2) + \cdots + E(u_5)$$

$$= \frac{1}{5} + \frac{1}{5} + \cdots + \frac{1}{5} = 1$$

$n = 5$

$$T_1 = \begin{cases} 1 & \text{if person \# 1 is in a tricycle} \\ 0 & \text{otherwise} \end{cases}$$

$P(T_1 = 1)$

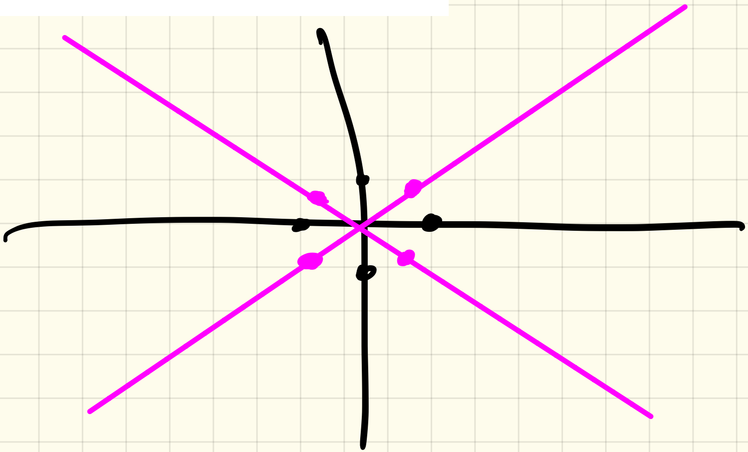
$(1 \; x \; y) \; \_ \; \_$

x: 4, y: 3, remaining: $2!$

$4!$ ways out of $5!$ perm

$$P(T_1 = 1) = \frac{4!}{5!} = \frac{1}{5}$$

$$u_{2n} = \frac{1}{6^{2n}} \sum_{j,k} \frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!}$$

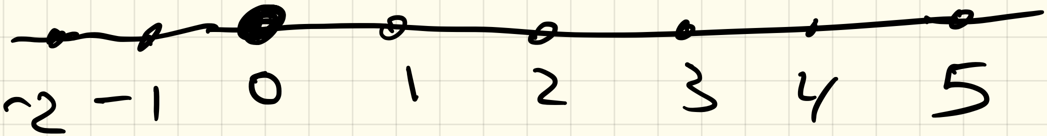


$$T := T_1 + T_2 + T_3 + T_4 + T_5$$

$$= 3 \cdot \#\text{ of tricycles}$$

$$E(T) = 1$$

$$E(\#\text{ of tricycles}) = \frac{1}{3}$$



$u := P(\text{starting at } 0\text{, we return to } 0)$

$E(\#\ \text{returns})$

$P(\text{return exactly } k \text{ time})$

$u^k(1-u)$

E(# of returns)

$$= 1 \cdot P(\text{Ret } 1) + 2\, P(\text{Return } 2) + \cdots$$

$$= \sum_{k=1}^{\infty} k u^k (1-u)$$

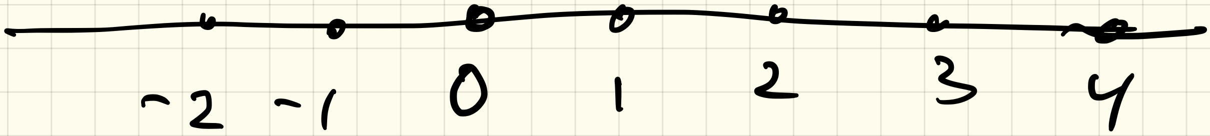
$$= 1 \cdot u(1-u) + 2 \cdot u^2 (1-u) + 3 \cdot u^3 (1-u) + \cdots$$

$$u - u^2 + 2u^2 - 2u^3 + 3u^3 - 3u^4 + \cdots$$

$$= u + u^2 + u^3 + \cdots = \frac{u}{1-u}$$

If $P(\text{Return} = 1) \iff$

$E(\#\text{ returns}) = \infty$



$u_t := \begin{cases} 1 & \text{if we are back at } 0 \text{ at time } t \\ 0 & \text{otherwise} \end{cases}$

$E(u_2 + u_4 + u_6 + u_8 + \dots)$

$= P(u_2 = 1) + P(u_4 = 1) + \cdots$

$$P(U_{10}=1) = \frac{\binom{10}{5}}{2^{10}} = \frac{\frac{10!}{5!5!}}{2^{10}}$$

all we need to is

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}}$$

$n=5$

$$P(u_{10}=1) = \binom{10}{5}\frac{1}{2^{10}}$$

$$\frac{10!}{5!\,5!\,2^5\,2^5} = \frac{1\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot 10}{2\,4\,6\,8\,10\quad 2\,4\,6\,8\,10}$$

$$= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\;8\;10} = X$$

$$\underbrace{\frac{1}{2}\;\frac{3}{4}\;\frac{5}{6}\;\frac{7}{8}\;\frac{9}{10}}_{X}\quad \underbrace{\frac{1}{2}\;\frac{2}{3}\;\frac{4}{5}\;\frac{6}{7}\;\frac{8}{9}}_{X'} = \frac{1}{20}$$

$$\frac{1}{20} < x^2$$

$n = 5$ $$\frac{1}{\sqrt{20}} < x$$

$n = 4$ $$\frac{1}{\sqrt{2.8}} < x$$

$$\frac{\binom{2n}{n}}{2^{2n}} > \frac{1}{\sqrt{4n}} = \frac{1}{2\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} > \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \frac{1}{2}\sum \frac{1}{n}$$

$$= \infty$$

$$\frac{1}{2\sqrt{n}} < \frac{\binom{2n}{n}}{2^{2n}} < \frac{1}{\sqrt{n}}$$

$$\frac{\binom{2n}{n}}{2^{2n}} \approx \frac{1}{\sqrt{\pi n}}$$