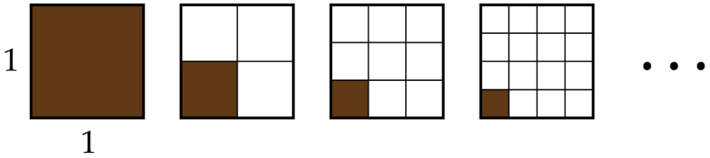
Euler Strikes Again

MATHπATH 2018 • LEWIS & CLARK COLLEGE

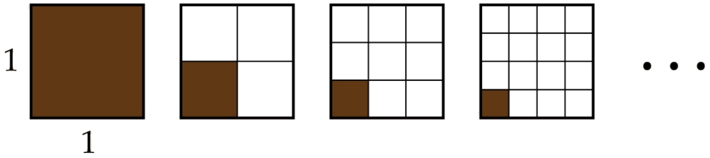
Sam Vandervelde • Proof School • July 4, 2018

Find That Formula



What is the total area of the shaded squares?

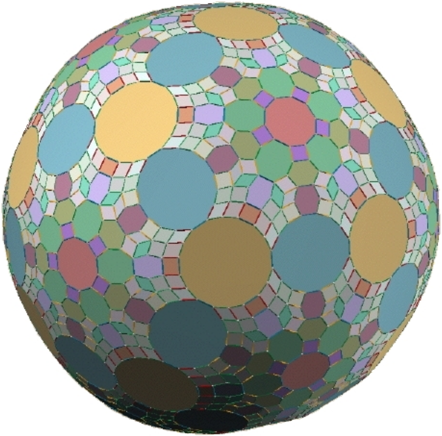
Find That Formula



What is the total area of the shaded squares?

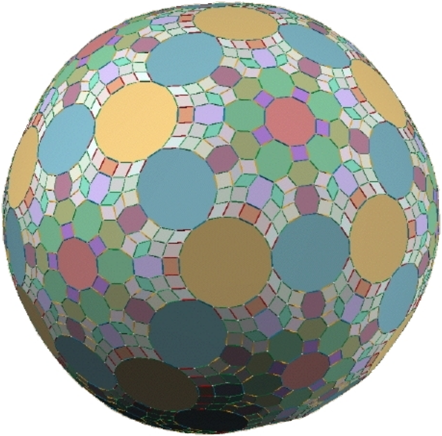
$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^{2}}{6}.$$

Find That Formula



Let V, E, and F be the total number of vertices, edges and faces in the polyhedron shown at left. Then we have

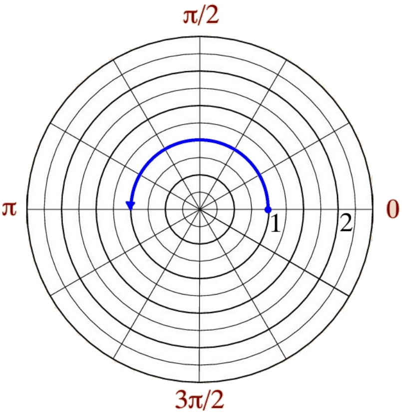
Find That Formula



Let V, E, and F be the total number of vertices, edges and faces in the polyhedron shown at left. Then we have

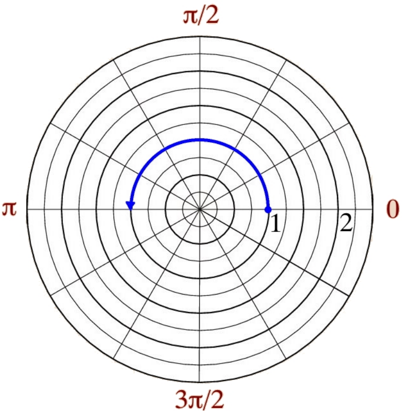
$$V - E + F = \mathbf{2}.$$

Find That Formula



The path traced out on this polar graph paper gives a nice visual representation of what famous formula relating i, π, e?

Find That Formula



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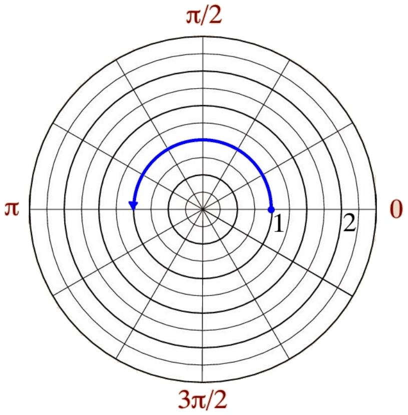
$$\pi i e = \text{yummy}.$$

Find That Formula



The path traced out on this polar graph paper gives a nice visual representation of what famous formula relating i, π, e?

$$e^{\pi i} = -1.$$

Attribution Please

The three formulas just presented are favorites of mathematicians everywhere. Which one is due to Euler?

A. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$

B. $V - E + F = 2$

C. $e^{\pi i} = -1$

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B. $V - E + F = 2$

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They all are!

And Now For Something Completely Different

Imagine that during a mind-numbing plenary session you were to use the `RAND` feature on your calculator to generate a long list of random numbers, each between 0 and 1.

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Suppose that you decided to begin adding up these random numbers.

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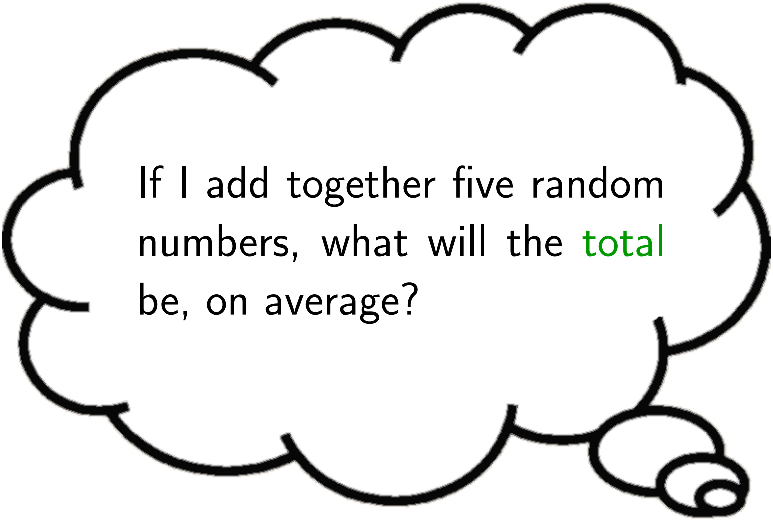
Imagine that during a mind-numbing plenary session you were to use the RAND feature on your calculator to generate a long list of random numbers, each between 0 and 1.

Suppose that you decided to begin adding up these random numbers.

Perhaps you begin to speculate about the sorts of sums that arise in this manner...

An Obvious Question



An Obvious Question

If I add together five random numbers, what will the total be, on average? Surely 2.5.

A Not So Obvious Question

What is the probability that the sum of the five random numbers lies between 2 and 3?

A Not So Obvious Question

What is the probability that the sum of the five random numbers lies between 2 and 3?

Hmmm. . . that's harder.

Democratic Math

Let's take a vote. Would you say that the probability that the sum lies between 2 and 3

A. is less than 50%

B. is equal to 50%

C. is greater than 50%

Democratic Math

Let's take a vote. Would you say that the probability that the sum lies between 2 and 3

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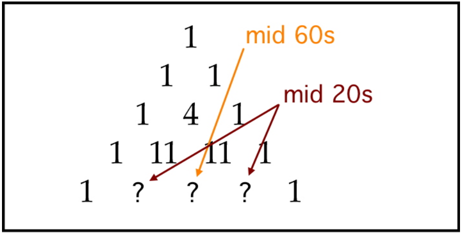
C. is greater than 50%

Time to find out!

A Nifty Triangle of Numbers

Here are the numbers we've found so far:



What are the mystery values? DISCUSS

A Nifty Triangle of Numbers

Here are the numbers we've found so far:

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

⋆ Has anyone seen these numbers before?

A Nifty Triangle of Numbers

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				1				
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- ⋆ Has anyone seen these numbers before?
- ⋆ What do they remind you of?

A Nifty Triangle of Numbers

Here are the numbers we've found so far:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1
\end{array}
$$

- ⋆ Has anyone seen these numbers before?
- ⋆ What do they remind you of?
- ⋆ Can you find any neat properties?

A Nifty Triangle of Numbers

Here are the numbers we've found so far:

	row sum
1	1
1 1	2
1 4 1	6
1 11 11 1	24
1 26 66 26 1	120

Very good; the row sums give factorials.

A Nifty Triangle of Numbers

Here are the numbers we've found so far:

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Very good; the row sums give factorials.
Now we'll learn how to generate each row from the previous one. Can you guess?

Row Generation

We'll illustrate the technique to obtain row 5.

$n=4$	1	11	11	1
$n=5$	?	?	?	

Row Generation

We'll illustrate the technique to obtain row 5.

	5	**1**				
n=4	0	1	11	11	1	0
n=5						

We include a few auxiliary numbers,

Row Generation

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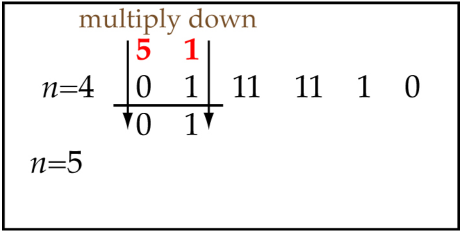

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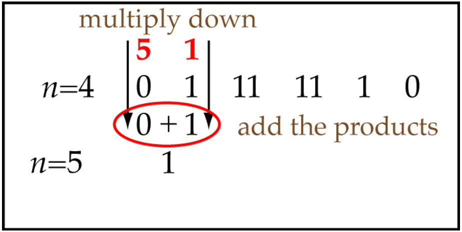


We include a few auxiliary numbers, multiply, and add the results.

Row Generation

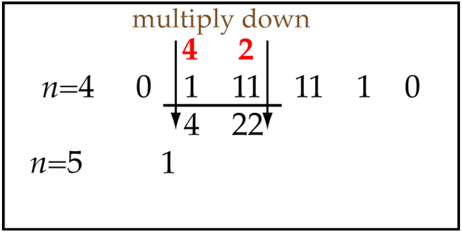
We'll illustrate the technique to obtain row 5.

		4	**2**			
n=4	0	1	11	11	1	0
n=5		1				

The auxiliary numbers change as we move across,

Row Generation

We'll illustrate the technique to obtain row 5.



The auxiliary numbers change as we move across, but the rest stays the same.

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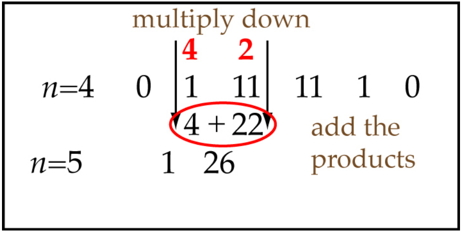

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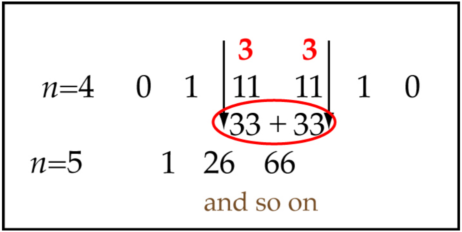

Repeat this process to obtain the entire row.

Row Generation

Now you use this technique to obtain row 6.

	6	**1**				
$n=5$	0	1	26	66	26	1
$n=6$						

The first few auxiliary numbers are shown to help you get started. *Try it now!*

Row Generation

Now you use this technique to obtain row 6.

n=5	1	26	66	26	1	
n=6	1	57	302	302	57	1

If all went well, you computed the row of six numbers displayed above.

Vandervelde Numbers

Here's the entire triangle so far.

					1					
				1		1				
			1		4		1			
		1		11		11		1		
	1		26		66		26		1	
1		57		302		302		57		1

Vandervelde Numbers

Here's the entire triangle so far.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 4 & & 1 & & & \\ & & 1 & & 11 & & 11 & & 1 & & \\ & 1 & & 26 & & 66 & & 26 & & 1 & \\ 1 & & 57 & & 302 & & 302 & & 57 & & 1 \end{array}$$

I discovered these numbers, so I get to name them.

Vandervelde Numbers

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					1					
				1		1				
			1		4		1			
		1		11		11		1		
	1		26		66		26		1	
1		57		302		302		57		1

I discovered these numbers, so I get to name them. I'll call them Vandervelde numbers.

Scooped By Euler

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1.2.3(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1.2.3.4(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1.2.3.4.5(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1.2.3.4.5.6(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1.2.3.4.5.6.7(p-1)^7}$$

&c.

L. Euler, 1755.

Eulerian Polynomials

$\frac{A_n(p)/p}{n!(p-1)^n} \quad (1 \le n \le 7)$

Guess who began studying the properties of these numbers over 250 years ago?

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$$\alpha = \frac{1}{1(p-1)}$$
$$\beta = \frac{p+1}{1.2(p-1)^2}$$
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&c.

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Eulerian Polynomials

$$\frac{A_n(p)/p}{n!(p-1)^n} \quad (1 \le n \le 7)$$

I guess we'll have to call them by their proper name, Eulerian numbers.

Notable Notation

row n						
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

In order to easily refer to a particular Eulerian number, we'll indicate its row

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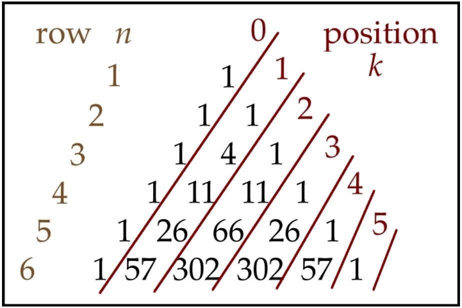
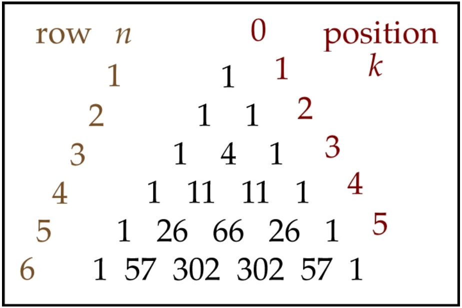


In order to easily refer to a particular Eulerian number, we'll indicate its row and its position within that row.

Notable Notation



The Eulerian number in row n and position k is variously called $A(n,k)$ or $E(n,k)$ or $\left\langle {n \atop k} \right\rangle$.

Notable Notation

row n							position k
							0
1	1						1
2	1	1					2
3	1	4	1				3
4	1	11	11	1			4
5	1	26	66	26	1		5
6	1	57	302	302	57	1	

I prefer $\left\langle {n \atop k} \right\rangle$, so that's what we'll use today.

Notable Notation

row n / position k	0	1	2	3	4	5
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

I prefer $\left\langle {n \atop k} \right\rangle$, so that's what we'll use today.
Thus $\left\langle {3 \atop 1} \right\rangle = 4$ and $\left\langle {6 \atop 4} \right\rangle = 57$.

Deft Definitions

Let's summarize what we know about Eulerian numbers, using our new notation.

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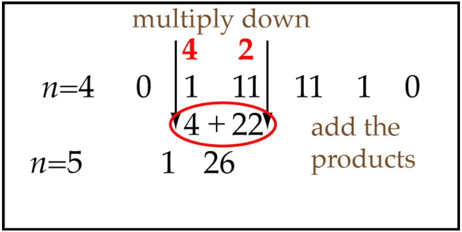
$$\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 11 & 11 & 1 & 0 & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & 0 \end{array}$$

We define $\left\langle {1 \atop 0} \right\rangle = 1$ and $\left\langle {1 \atop k} \right\rangle = 0$ for $k \neq 0$; this gives the complete first row of the table.

Deft Definitions



We then declare that

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle.$$

This gives all subsequent rows of the table.

Pretty Properties

	row sum
1	1
1 1	2
1 4 1	6
1 11 11 1	24
1 26 66 26 1	120

We have discovered that

$$\left\langle {n \atop 0} \right\rangle + \left\langle {n \atop 1} \right\rangle + \cdots + \left\langle {n \atop n-1} \right\rangle = n!.$$

(Why does this make sense, by the way?)

Our First Result

Finally, Eulerian numbers provide the answer to our probability question.

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THEOREM 1: Suppose we choose n real numbers at random, each between 0 and 1. Then the probability that their sum lies between k and $k+1$ is given by $\frac{1}{n!}\left\langle {n \atop k} \right\rangle$.

Our First Result

Finally, Eulerian numbers provide the answer to our probability question.

THEOREM 1: Suppose we choose n real numbers at random, each between 0 and 1. Then the probability that their sum lies between k and $k+1$ is given by $\frac{1}{n!}\left\langle {n \atop k} \right\rangle$.

PROOF: *Stay tuned!*

They're Everywhere

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

You didn't realize it, but Eulerian numbers crop up all over the place.

They're Everywhere

1					
1	1				
1	4	1			
1	11	11	1		
1	26	66	26	1	
1	57	302	302	57	1

See if you can spot the Eulerian numbers in the following slides. (Raise your hand when you think you've found one.)

They're Everywhere

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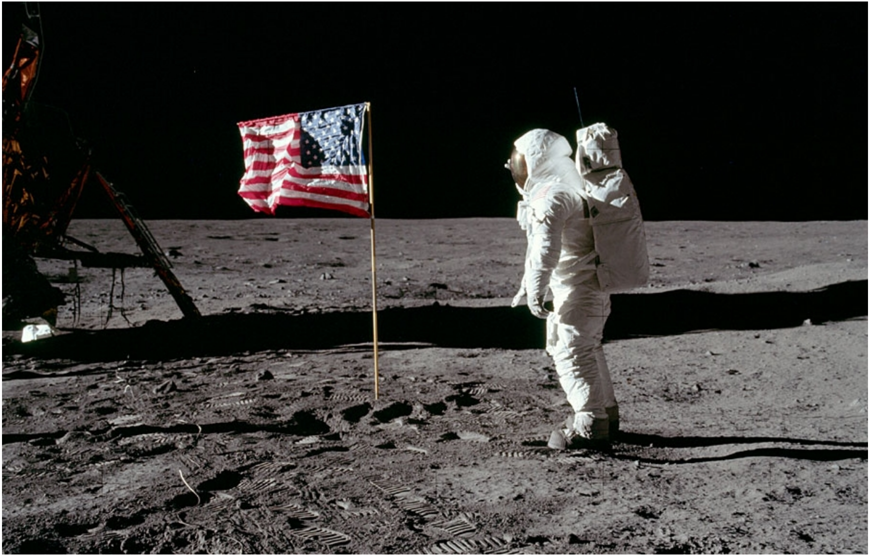

They're Everywhere

They're Everywhere

They're Everywhere

Right, a marathon is 26 miles long.

Counting Ascents

Consider the following permutation of the numbers 1, 2 and 3:

2 3 1

Counting Ascents

Consider the following permutation of the numbers 1, 2 and 3:

$$2 \nearrow 3 \searrow 1$$

This permutation involves one ascent and one descent.

Counting Ascents

Let's count the total number of ascents for each permutation of 1, 2, 3 just for kicks.

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- 1 2 3

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- 1 2 3 2 ascents
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- 1 2 3 2 ascents
- 1 3 2 1 ascent
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- 2 1 3 1 ascent
- 2 3 1 1 ascent
- 3 0 2 1 ascent
- 3 2 1 0 ascents

Counting Ascents

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- 1 2 3 — 2 ascents
- 1 3 2 — 1 ascent
- 2 1 3 — 1 ascent
- 2 3 1 — 1 ascent
- 3 0 2 — 1 ascent
- 3 2 1 — 0 ascents

Salient alien

In general, if we sort permutations according to the number of ascents, the sizes of the resulting sets are given by Eulerian numbers.

Salient alien

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Roller Coaster Math

Consider a permutation of 1, 2, 3, 4, 5:

$$* \nearrow * \searrow * \nearrow * \searrow *$$

Roller Coaster Math

Consider a permutation of 1, 2, 3, 4, 5:

$$* \nearrow * \searrow * \nearrow * \searrow *$$

$$\uparrow$$

What if we insert a 6 right there? How will the number of ascents change?

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How many different locations are there to insert a 6, while we're at it?

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$$* \nearrow * \searrow * \nearrow * \searrow *$$
$$\uparrow$$

What if we insert a 6 right there? How will the number of ascents change? *Not at all*

How many different locations are there to insert a 6, while we're at it? **6**

Roller Coaster Math

Consider a permutation of 1, 2, 3, 4, 5:

$$\uparrow \quad * \nearrow * \searrow * \nearrow * \searrow *$$

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Consider a permutation of 1, 2, 3, 4, 5:

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$\uparrow$

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Roller Coaster Math

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What if we insert a 6 right there? How will the number of ascents change?

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Euler Would Be Proud

Let's build a permutation of 1, 2, 3, 4, 5, 6 having precisely three ascents, by inserting a 6 into permutations of 1, 2, 3, 4, 5.

$* \quad * \quad * \quad * \quad *$ $\qquad\qquad$ # ascents?

Euler Would Be Proud

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$$* \quad * \quad * \quad * \quad * \qquad \nearrow \; \nearrow \; \searrow \; \searrow$$

How many ways to insert the 6?

Euler Would Be Proud

Let's build a permutation of 1, 2, 3, 4, 5, 6 having precisely three ascents, by inserting a 6 into permutations of 1, 2, 3, 4, 5.

$$* \quad * \quad * \quad * \quad * \qquad \nearrow \; \nearrow \; \searrow \; \searrow$$

How many ways to insert the 6? **3**

So far we have $3(\#\text{ ways ascent} = 2)$.

Euler Would Be Proud

Let's build a permutation of 1, 2, 3, 4, 5, 6 having precisely three ascents, by inserting a 6 into permutations of 1, 2, 3, 4, 5.

$$* \quad * \quad * \quad * \quad * \qquad \nearrow \; \nearrow \; \nearrow \; \searrow$$

Now how many ways to insert the 6?

Euler Would Be Proud

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$$* \quad * \quad * \quad * \quad * \qquad \nearrow \; \nearrow \; \nearrow \; \searrow$$

Now how many ways to insert the 6? **4**

This gives $4(\#\text{ ways ascent} = 3)$.

Euler Would Be Proud

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The same reasoning works in general. *AYD*

Our Second Result

THEOREM 2: Let n be a positive integer, so there are $n!$ permutations of the numbers 1, 2, 3, . . . , n. For each integer k in the range $0 \leq k \leq n-1$ there are a total of $\left\langle {n \atop k} \right\rangle$ permutations having exactly k ascents.

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Slicing a Cube

- Let's return to our probability question for a moment and consider it from a geometric vantage point.

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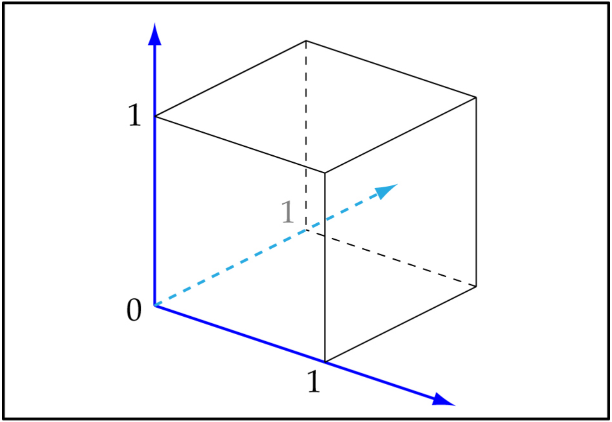
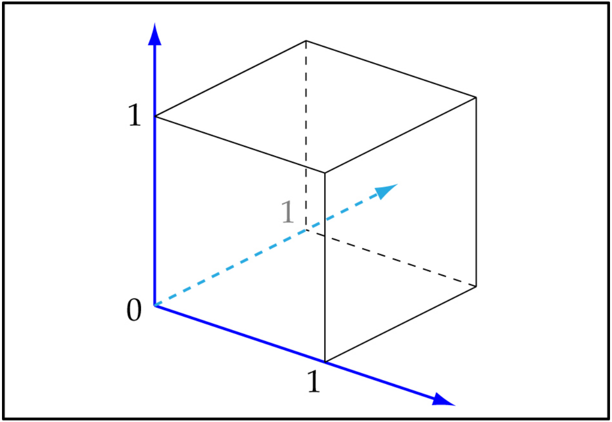
- Let's return to our probability question for a moment and consider it from a geometric vantage point.
- What is a convenient way to represent the set of all possible triples of numbers, each of which lies between 0 and 1?

Slicing a Cube

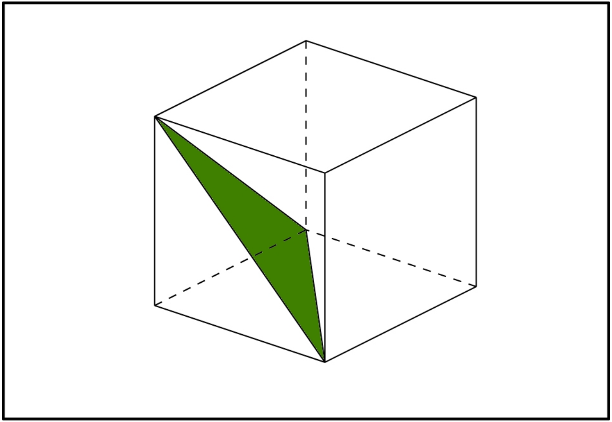


Slicing a Cube



Which portion corresponds to the triples of numbers whose sum is between 0 and 1?

Slicing a Cube



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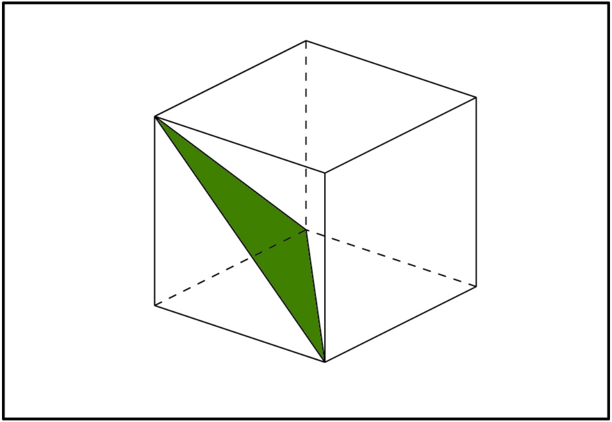
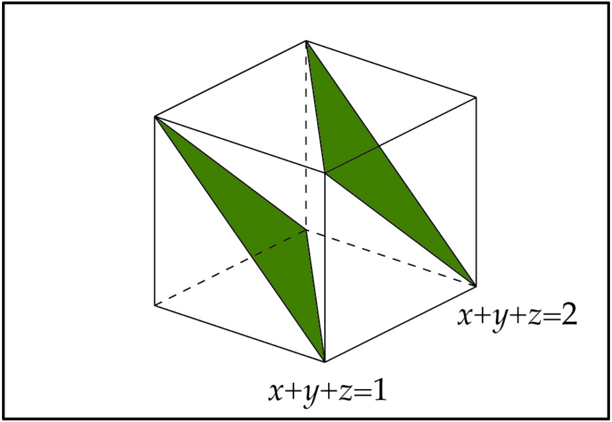


Which portion corresponds to the triples of numbers whose sum is between 1 and 2?

Slicing a Cube



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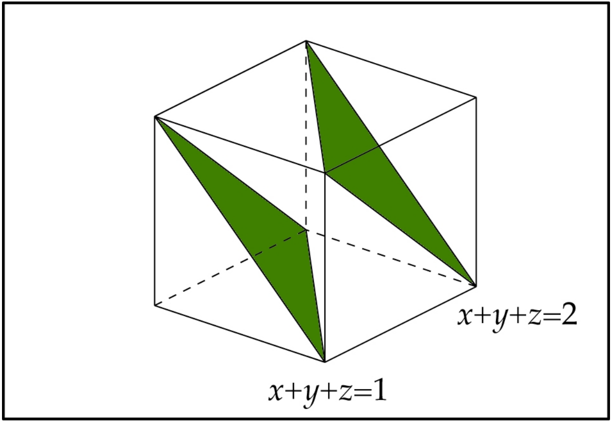
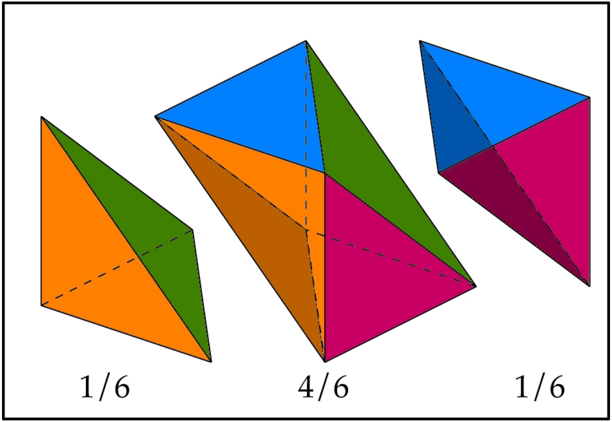


How do we obtain the probabilities from this geometric diagram?

Slicing a Cube



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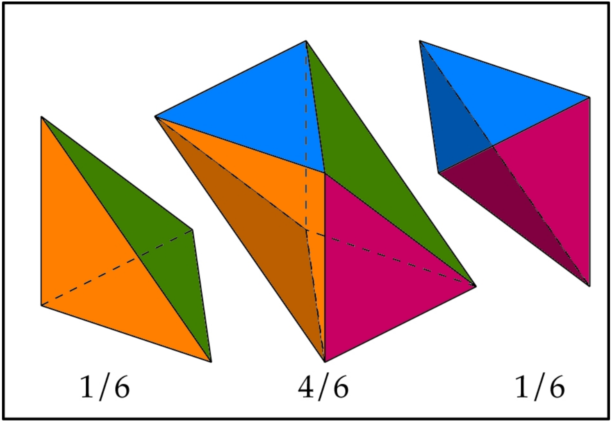

Guess what happens when we add four or five random numbers rather than three?

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- Carve up the unit 4-cube into regions such as $x < z < y < w$.
- How many such regions are there? **24**
- How do their volumes compare?

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- Carve up the unit 4-cube into regions such as $x < z < y < w$.
- How many such regions are there? **24**
- How do their volumes compare? **equal**

Is It Lunch Time Yet?

There is a clever way to transform each region like $x < z < y < w$ into a new one:

$$Z = z$$

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Y &= y - z
\end{aligned}
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4	1	3	2

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Observe that $W + X + Y + Z = 1 + w$, therefore $1 < W + X + Y + Z < 2$.

I'm Really Hungry

Let's try out another such region, transforming $w < z < x < y$ into a new region:

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I'm Really Hungry

Let's try out another such region, transforming $w < z < x < y$ into a new region:

$$Z = z$$
$$Y = y - z$$
$$X = x - y + 1$$
$$W = w - x + 1$$

w	x	y	z
1	3	4	2

Observe that $W + X + Y + Z = 2 + w$, therefore $2 < W + X + Y + Z < 3$.

Give Me Food Now

Just to move things along, we'll employ a QuickProof$^{\text{TM}}$.

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- Volume is preserved.

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- Volume is preserved. Routine
- Regions don't overlap.

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Just to move things along, we'll employ a QuickProof^{TM}. What would it take for this strategy to work?

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- Regions don't overlap. No problem
- You're still awake.

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Just to move things along, we'll employ a QuickProof™. What would it take for this strategy to work?

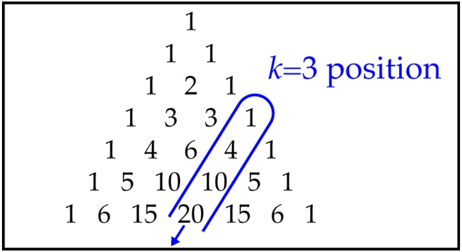
- Volume is preserved. Routine
- Regions don't overlap. No problem
- You're still awake. WELL!?!?

Voilá, an elegant proof of the random number sum conjecture is ours.

Public Service Announcement

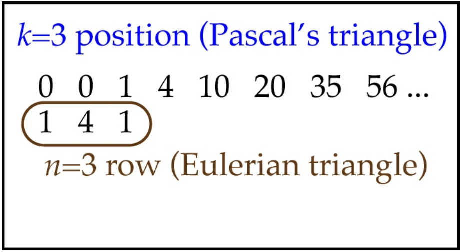
We're going to speed up a little bit now. You may want to fasten your seatbelts...

More Power To You



Here is a nifty surprise. Look at the numbers in the $k = 3$ position within Pascal's triangle.

More Power To You



Now we line up the $n = 3$ row of Eulerian numbers alongside them.

More Power To You

0	0	1	4	10	20	35	56 ...
1	4	1					
0	0	1					

Next follow the usual routine: multiply down,

More Power To You

$$\begin{array}{llllllll} 0 & 0 & 1 & 4 & 10 & 20 & 35 & 56 \ldots \\ 1 & 4 & 1 \\ \hline \end{array}$$
$$0+0+1 = \mathbf{1}$$

Next follow the usual routine: multiply down, then add across.

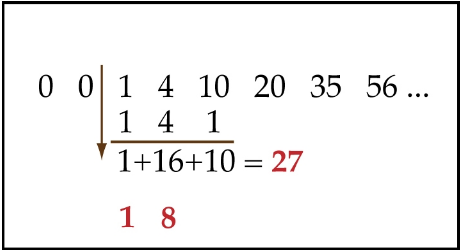
More Power To You

0 | 0 1 4 10 20 35 56 ...
1 4 1
0 + 4 + 4 = 8

1

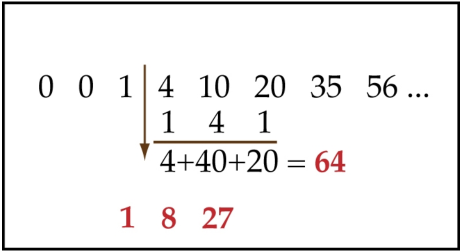
Repeat this process until the pattern appears.

More Power To You



Do you see it yet?

More Power To You



Do you see it yet? Isn't that amazing?

More Power To You

0	0	1	4	10	20	35	56 ...
			1	4	1		
		1	8	27	64		

What happens if we overlay the fourth row of the Eulerian triangle with the numbers in the $k = 4$ position within Pascal's triangle?

More Power To You

0 0 1 4 10 20 35 56 ...

1 4 1

1 8 27 64

What happens if we overlay the fourth row of the Eulerian triangle with the numbers in the $k = 4$ position within Pascal's triangle? *Yup*

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PROOF: by story. (Euler is delighted!)

Proof By What!?

Here's how a combinatorial proof works.

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Proof By GATTLP

Here's how a combinatorial proof works.

- Next $\binom{n+2}{4}\left\langle {4 \atop 2} \right\rangle$: $a \quad b \quad c \quad \cdots \quad n \quad = \quad \doteq$

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- What to do about three equal signs?

$$[\ a = \doteq \ddot{=}\ ] \longrightarrow$$

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$$[\; a = \doteq b \;] \longrightarrow aabb?$$

- What to do about three equal signs?

$$[\; a = \doteq \ddot{=} \;] \longrightarrow aaaa$$

Proof By SWATSD

If you can find just the right way to interpret equal signs, then you will have a beautiful proof of this identity!

G O O D L U C K

Power Play

We have just seen that by combining Eulerian numbers with Pascal's triangle, we can produce perfect powers.

Power Play

We have just seen that by combining Eulerian numbers with Pascal's triangle, we can produce perfect powers.

But did you know that by taking differences of perfect powers, we can recover the Eulerian numbers?

Check it out:

Power Play

0	0	0	0	1	8	27	64	125

Power Play

0	0	0	0	1	8	27	64	125

0	0	0	1	7

Power Play

0	0	0	0	1	8	27	64	125

0	0	0	1	7	19

Power Play

0	0	0	0	1	8	27	64	125
0	0	0	1	7	19	37		

Power Play

0	0	0	0	1	8	27	64	125
0	0	0	1	7	19	37	61	

Power Play

0		0		0		0		1		8		27		64		125
	0		0		0		1		7		19		37		61	
0		0		0		1										

Power Play

0		0		0		0		1		8		27		64		125
	0		0		0		1		7		19		37		61	
0		0		0		1		6		12		18		24		30

Power Play

0		0		0		0		1		8		27		64		125
	0		0		0		1		7		19		37		61	
0		0		0		1		6		12		18		24		30
	0		0		1		5		6		6		6		6	

Power Play

0		0		0		0		1		8		27		64		125
	0		0		0		1		7		19		37		61	
0		0		0		1		6		12		18		24		30
	0		0		1		5		6		6		6		6	
0		0		1		4		1		0		0		0		0

Pretty sweet, wouldn't you say?

Our Fifth Result

Naturally, if we began with fourth powers, we would have obtain the fourth row of the Eulerian triangle instead.

Our Fifth Result

Naturally, if we began with fourth powers, we would have obtain the fourth row of the Eulerian triangle instead. In general, we have

THEOREM 5: Create an infinite sequence of integers consisting of all 0's followed by the perfect n^{th} powers. After taking $n+1$ differences, we will be left with a row of all 0's except for row n of the Eulerian triangle.

Our Fifth Result

Here's what the diagram looks like in general:

$$0 \quad 0 \quad 0 \quad 0 \quad 1^n \quad 2^n \quad 3^n \quad 4^n \quad 5^n$$

differences go here

$$0 \quad 0 \quad \left\langle {n \atop 0} \right\rangle \quad \left\langle {n \atop 1} \right\rangle \quad \left\langle {n \atop 2} \right\rangle \quad \cdots \quad \left\langle {n \atop n-1} \right\rangle \quad 0$$

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Try $n = 4$ for fun, or $n = 5$ if you dare.

That's All Folks

The  End

Thanks for being such a great audience!