≡ Neutron Graphs ≡

MATHπATH 2018 • LEWIS & CLARK COLLEGE

Sam Vandervelde • Proof School • July 5, 2018

Don't Fall For It

Which is heavier, a pound of feathers or a pound of lead?

Don't Fall For It

Which is heavier, a pound of feathers or a pound of lead?

=

They both weigh one pound, of course!

As Dense As It Gets

What volume of neutron star material has the same mass as a cube of rock 10 km (about 6 miles) on a side?

As Dense As It Gets

What volume of neutron star material has the same mass as a cube of rock 10 km (about 6 miles) on a side?

Polynomial Refresher

The degree of a polynomial refers to the highest power of x in its equation.

Polynomial Refresher

The degree of a polynomial refers to the highest power of x in its equation. What is the degree of each of these polynomials?

- $x^5 + x^2 - x$

Polynomial Refresher

The degree of a polynomial refers to the highest power of x in its equation. What is the degree of each of these polynomials?

- $x^5 + x^2 - x$ **5**
- $(x^5 + 1)/(x + 1)$

Polynomial Refresher

The degree of a polynomial refers to the highest power of x in its equation. What is the degree of each of these polynomials?

- $x^5 + x^2 - x$ **5**
- $(x^5 + 1)/(x + 1)$ **4**
- $2018x^4 - 1776x^3 + 1000x^7 - 5$

Polynomial Refresher

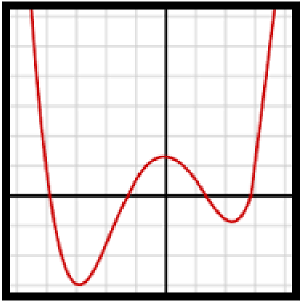
The degree of a polynomial refers to the highest power of x in its equation. What is the degree of each of these polynomials?

- $x^5 + x^2 - x$ **5**
- $(x^5 + 1)/(x + 1)$ **4**
- $2018x^4 - 1776x^3 + 1000x^7 - 5$ **7**

The degree influences the shape of the graph.

Hint: Count The Roots

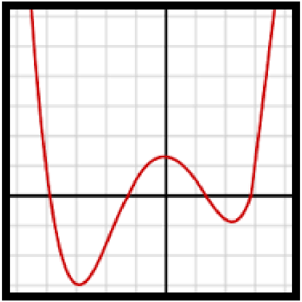
What degree polynomial is most likely to produce the following graph?



Hint: Count The Roots

What degree polynomial is most likely to produce the following graph?

VERY



GOOD

Coordinate Plane Hogs

Here is a degree five polynomial. It's going to require a lot of room to see all the interesting behavior!

$$x^5 - 10x^4 - 30x^2 + 40$$

Coordinate Plane Hogs

Here is a degree five polynomial. It's going to require a lot of room to see all the interesting behavior!

$$x^5 - 10x^4 - 30x^2 + 40$$

This degree six beast will be even worse.

$$32x^6 - 48x^4 + 18x^2 - 1$$

Specifications

A neutron graph is a polynomial graph that

1. Enters and exits the unit window at the corners, including the point $(1,1)$,

Specifications

A neutron graph is a polynomial graph that

1. Enters and exits the unit window at the corners, including the point $(1, 1)$,
2. has all local minimums and maximums along the border of the unit window,

Specifications

A neutron graph is a polynomial graph that

1. Enters and exits the unit window at the corners, including the point $(1, 1)$,
2. has all local minimums and maximums along the border of the unit window,
3. and exhibits no other noteworthy behavior outside the unit window.

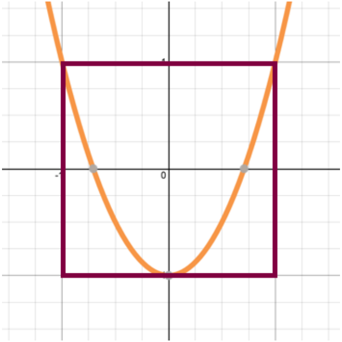
Specifications

A neutron graph is a polynomial graph that

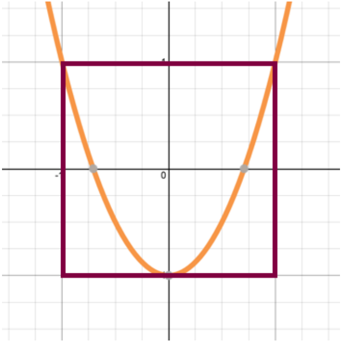
1. Enters and exits the unit window at the corners, including the point $(1, 1)$,
2. has all local minimums and maximums along the border of the unit window,
3. and exhibits no other noteworthy behavior outside the unit window.

How must a degree two neutron graph look?

Parabola Parameters

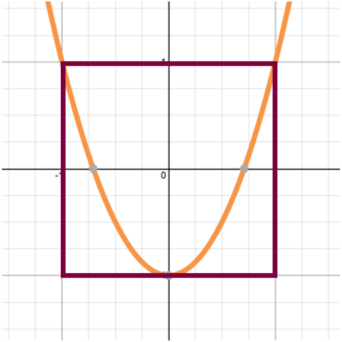


Parabola Parameters



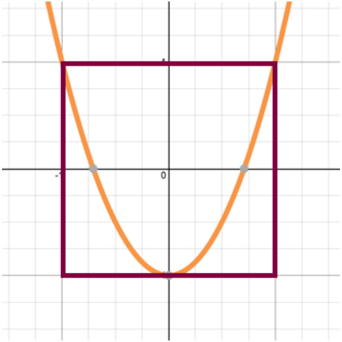
This parabola has an equation of the form $Ax^2 + Bx + C$. What are A, B and C?

Parabola Parameters



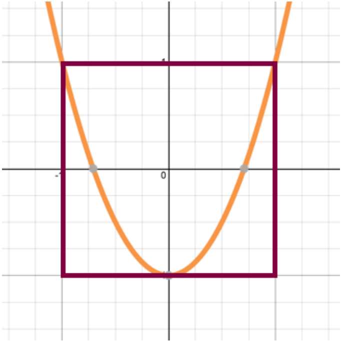
The y-intercept is at -1, so we may deduce that $C = -1$. Thus $y = Ax^2 + Bx - 1$.

Burbanas...



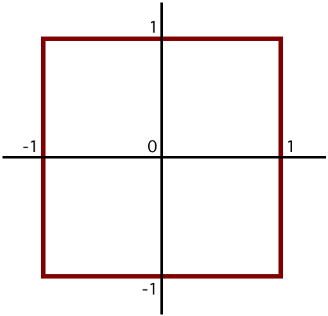
The graph is symmetric over the y-axis, hence $B = 0$. So far $y = Ax^2 - 1$.

Or Donucorns?



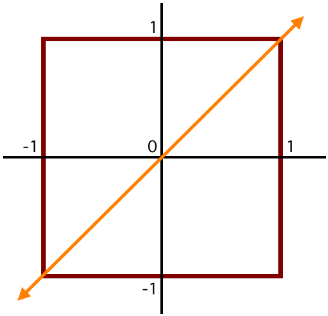
Finally, the graph passes through $(1,1)$, thus $A = 2$. In summary, we have $y = 2x^2 - 1$.

Low Hanging Fruit



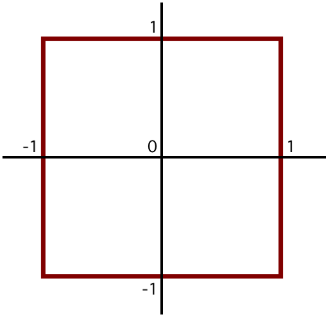
This will either seem easy or puzzling: what is the degree 1 neutron graph?

Low Hanging Fruit



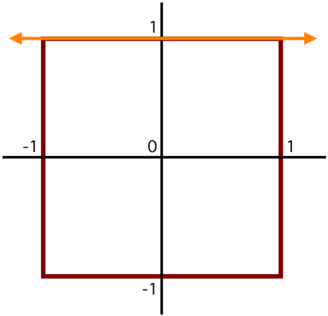
This will either seem easy or puzzling: what is the degree 1 neutron graph? $y = x$

Lower Hanging Fruit



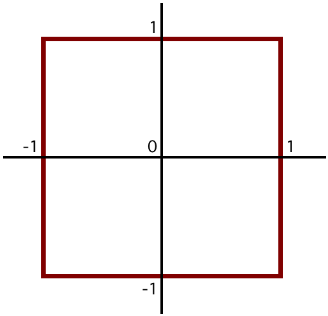
Even easier or more puzzling: what is the degree 0 neutron graph?

Lower Hanging Fruit



Even easier or more puzzling: what is the degree 0 neutron graph? $y=1$

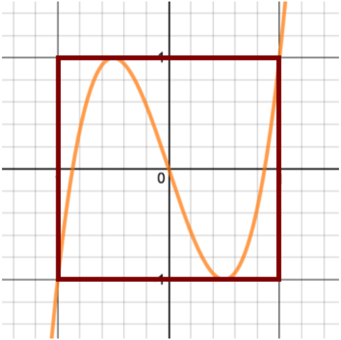
Cubic Conundrum



Next consider $y = Ax^3 + Bx^2 + Cx + D$.

How must a degree 3 neutron graph appear?

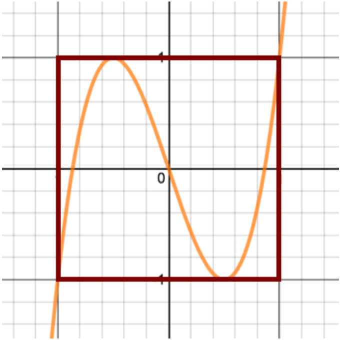
Cubic Conundrum



Next consider $y = Ax^3 + Bx^2 + Cx + D$.

How should we choose A, B, C and D?

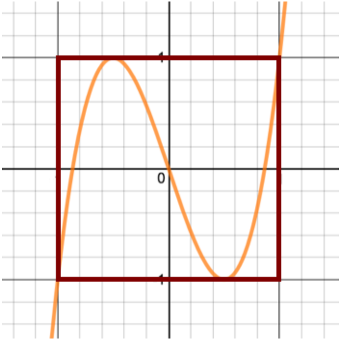
Burbanacorns!



The y-intercept idea reveals that $D = 0$.

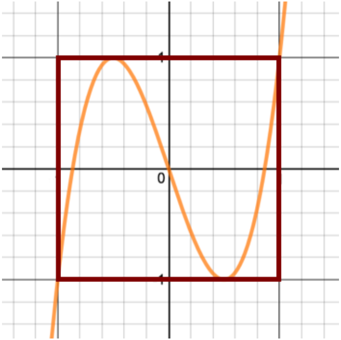
Now we have $y = Ax^3 + Bx^2 + Cx$.

Suburban Acorns?



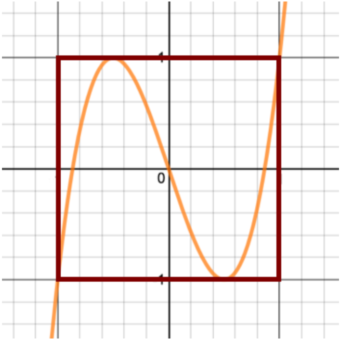
Symmetry about the origin suggests $B = 0$.
This brings us to $y = Ax^3 + Cx$.

Cubic Conundrum



The graph passes through $(1, 1)$, therefore $A + C = 1$, like $y = 2x^3 - x$ or $3x^3 - 2x$.

Cubic Zirconium



So our cubic graph must have the form $y = ax^3 - (a-1)x$. But how to choose a?

Barium Cobalt Nitrogen

Trying out values of a suggests that we select $a = 4$, so that $y = 4x^3 - 3x$. How can we confirm this choice?

Barium Cobalt Nitrogen

Trying out values of a suggests that we select $a = 4$, so that $y = 4x^3 - 3x$. How can we confirm this choice? *Calculus!*

Barium Cobalt Nitrogen

Trying out values of a suggests that we select $a = 4$, so that $y = 4x^3 - 3x$. How can we confirm this choice? *Calculus!*

- The slope is given by $12x^2 - 3$.

Barium Cobalt Nitrogen

Trying out values of a suggests that we select $a = 4$, so that $y = 4x^3 - 3x$. How can we confirm this choice? *Calculus!*

- The slope is given by $12x^2 - 3$.
- Low point requires $12x^2 - 3 =$

Barium Cobalt Nitrogen

Trying out values of a suggests that we select $a = 4$, so that $y = 4x^3 - 3x$. How can we confirm this choice? *Calculus!*

- The slope is given by $12x^2 - 3$.
- Low point requires $12x^2 - 3 = 0$.
- This occurs for $x =$

Barium Cobalt Nitrogen

Trying out values of a suggests that we select $a = 4$, so that $y = 4x^3 - 3x$. How can we confirm this choice? *Calculus!*

- The slope is given by $12x^2 - 3$.
- Low point requires $12x^2 - 3 = 0$.
- This occurs for $x = \frac{1}{2}$.
- The y-value there is...

Germanium Nickel Uranium Sulfur

Trying out values of a suggests that we select $a = 4$, so that $y = 4x^3 - 3x$. How can we confirm this choice? *Calculus*

- The slope is given by $12x^2 - 3$.
- Low point requires $12x^2 - 3 = 0$.
- This occurs for $x = \frac{1}{2}$.
- The y-value there is $y = -1$. *Nice!*

We now know the degree 3 neutron graph.

Making A List, Checking It Twice

Here are the formulas for our neutron graphs:

Degree	Formula
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$

Making A List, Checking It Twice

Here are the formulas for our neutron graphs:

Degree	Formula
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	???

What do you suppose is the formula the next polynomial in the list? DISCUSS

Making A List, Checking It Twice

Here are the formulas for our neutron graphs:

Degree	Formula
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4$

Making A List, Checking It Twice

Here are the formulas for our neutron graphs:

Degree	Formula
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 \qquad\qquad + 1$

Making A List, Checking It Twice

Here are the formulas for our neutron graphs:

Degree	Formula
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$

Let's test it out!

Neutron Patterns

Here's a productive perspective. Just like Fibonacci numbers, it's possible to build each neutron graph formula from the previous two. See if you can figure out how:

$$\begin{array}{r} (2x^2 - 1) \\ (4x^3 - 3x) \\ \hline 8x^4 - 8x^2 + 1 \end{array}$$

Neutron Patterns

Here's a productive perspective. Just like Fibonacci numbers, it's possible to build each neutron graph formula from the previous two. See if you can figure out how:

$$\begin{array}{r} (2x^2 - 1) \\ 2x(4x^3 - 3x) \\ \hline 8x^4 - 8x^2 + 1 \end{array}$$

Neutron Patterns

Here's a productive perspective. Just like Fibonacci numbers, it's possible to build each neutron graph formula from the previous two. See if you can figure out how:

$$\begin{array}{r} -(2x^2-1) \\ 2x(4x^3-3x) \\ \hline 8x^4-8x^2+1 \end{array}$$

That does the job quite nicely.

New Tron Pat Urns

What happens if we apply this technique on our degree three and four polynomials?

$$\begin{array}{r} -(4x^3 - 3x) \\ 2x(8x^4 - 8x^2 + 1) \\ \hline ? \qquad\qquad ? \end{array}$$

Neutron Patterns

What happens if we apply this technique on our degree three and four polynomials?

$$\begin{array}{r} -(4x^3 - 3x) \\ 2x(8x^4 - 8x^2 + 1) \\ \hline 16x^5 \qquad\qquad\qquad \end{array}$$

Neutron Patterns

What happens if we apply this technique on our degree three and four polynomials?

$$
\begin{array}{r}
-(4x^3 - 3x) \\
2x(8x^4 - 8x^2 + 1) \\
\hline
16x^5 - 20x^3 \qquad\qquad
\end{array}
$$

Neutron Patterns

What happens if we apply this technique on our degree three and four polynomials?

$$\begin{array}{r} -(4x^3 - 3x) \\ 2x(8x^4 - 8x^2 + 1) \\ \hline 16x^5 - 20x^3 + 5x \end{array}$$

Wanna see whether it works?

Neutron Patterns

What happens if we apply this technique on our degree three and four polynomials?

$$\begin{array}{r} -(4x^3 - 3x) \\ 2x(8x^4 - 8x^2 + 1) \\ \hline 16x^5 - 20x^3 + 5x \end{array}$$

Wanna see whether it works?

Looks like our work here is done.

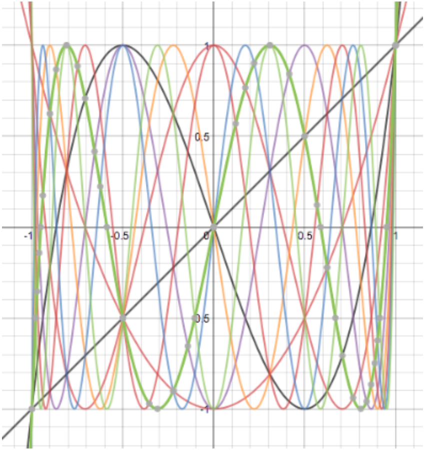
Making A List, Checking It Twice

Here are formulas for lots of neutron graphs:

Degree	Formula
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$
6	$32x^6 - 48x^4 + 18x^2 - 1$
7	$64x^7 - 112x^5 + 56x^3 - 7x$

Order Within Chaos

Here are the neutron graphs up to degree 9.



Trig To The Rescue

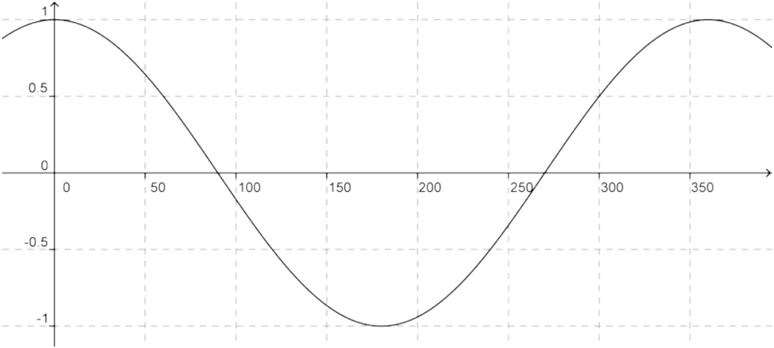
The key to unlocking the mystery of neutron graphs is cosine.

Trig To The Rescue

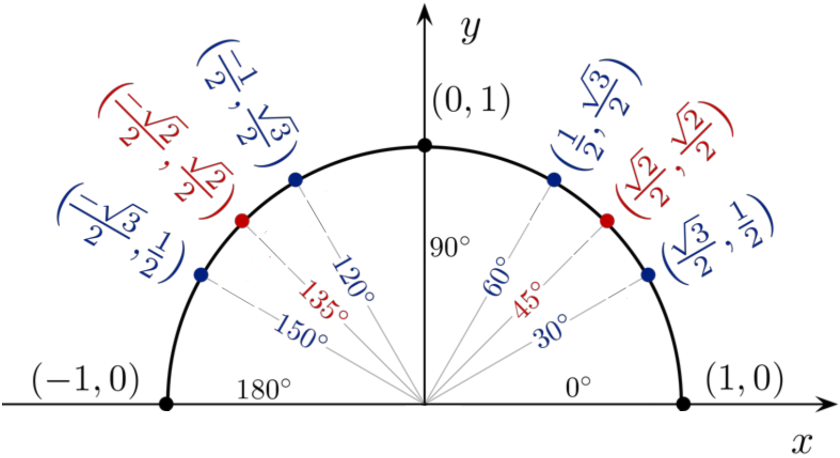
The key to unlocking the mystery of neutron graphs is cosine. *Yay for trigonometry!*

Trig To The Rescue

The key to unlocking the mystery of neutron graphs is cosine. Let's review a few basics.



Trig To The Rescue



Seeing Double

Here is a natural question to ask:

If we know the value of $\cos\theta$*, can we deduce the value of* $\cos(2\theta)$*?*

Seeing Double

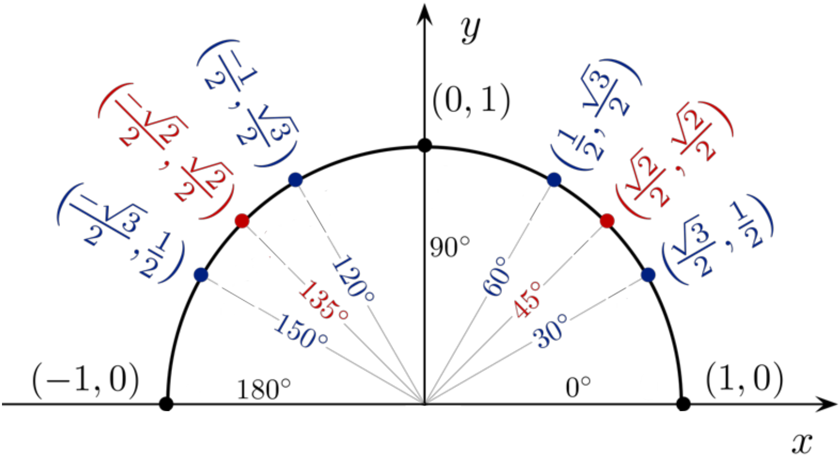
Here is a natural question to ask:

> *If we know the value of* $\cos\theta$*, can we deduce the value of* $\cos(2\theta)$*?*

For instance, suppose we know that $\cos\theta = \frac{1}{2}$. What is the value of $\cos(2\theta)$?

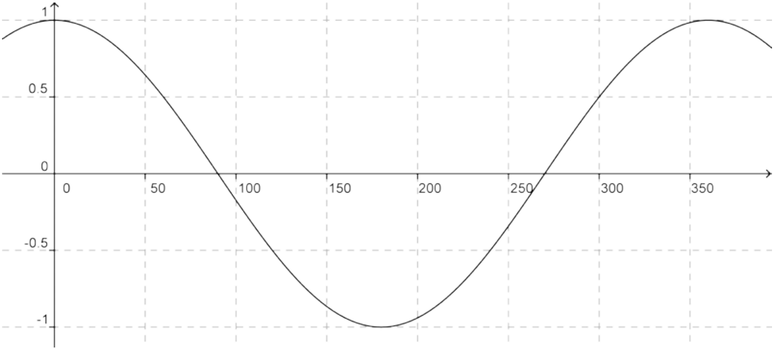
(Um, I forgot that chart already...)

Seeing Double



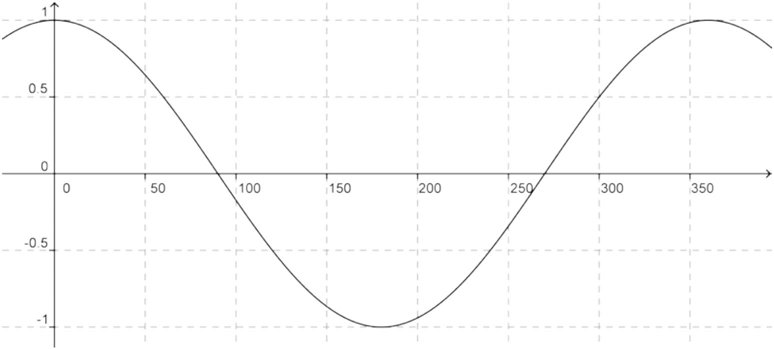
Seeing Double

What if $\cos\theta = -0.2$; estimate $\cos(2\theta)$.



Seeing Double

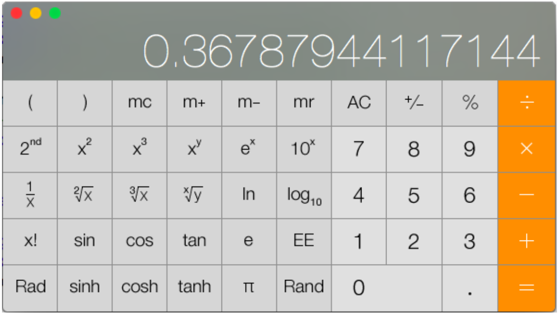
What if $\cos\theta = -0.2$; estimate $\cos(2\theta)$.



Very close! The precise answer is -0.92.

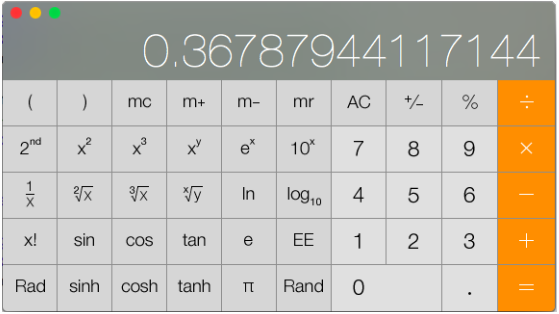
What Is The Difference

What if $\cos\theta = \frac{2}{3}$; calculate $\cos(2\theta)$.



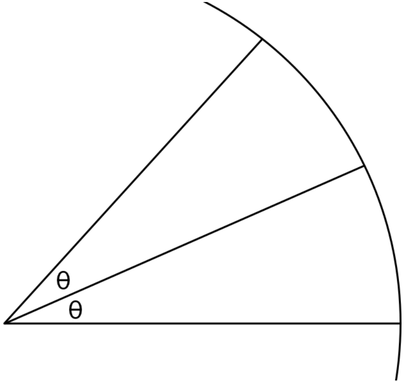
What Is The Difference

What if $\cos\theta = \frac{2}{3}$; calculate $\cos(2\theta)$.



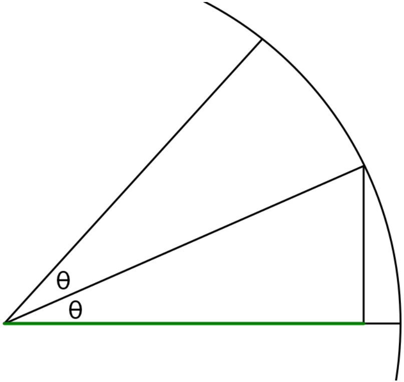
How suspicious; that is clearly $-\frac{1}{9}$.

Between Seal and Sea Lion?



It's time to settle this. Where is $\cos\theta$?

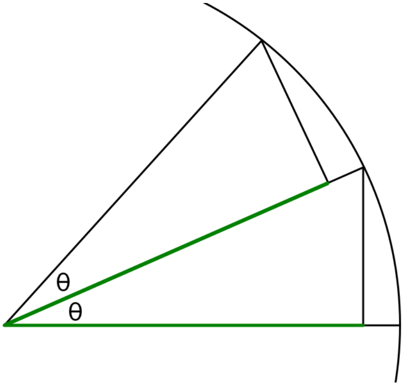
Answer: One Electron



It's time to settle this. Where is $\cos\theta$? *Good*

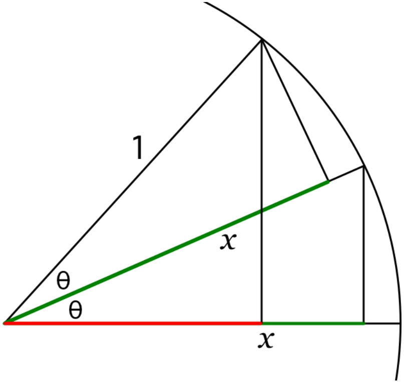
Where **else** is $\cos\theta$?

Double Vision



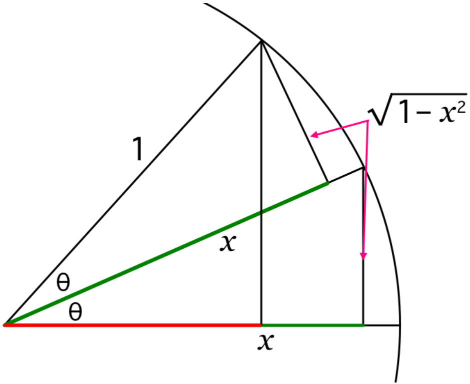
It's time to settle this. Where is $\cos\theta$? *Good*
Where **else** is $\cos\theta$? *Yes* Where is $\cos(2\theta)$?

Double Vision



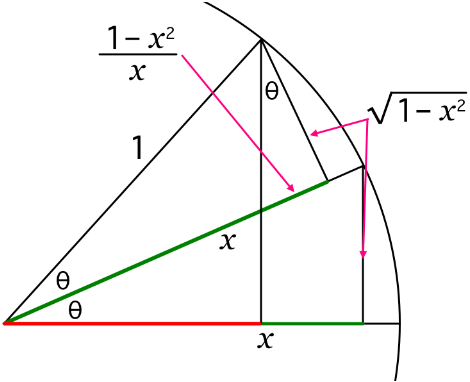
Let's label the green length as x; we want to find the red length in terms of x. What next?

Double Vision



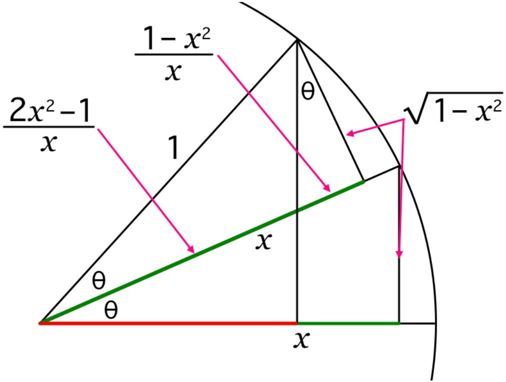
Let's hear it for Pythagoras. Can you put some similar triangles to work now?

This Is Too Phosphorous!



Not bad. We must be getting close! What do you suppose the next step is?

This Slide is Boron



Nice ALGEBRASKILLZ™. We're almost there. Can anyone bring us home?

All the Good Jokes Argon

Theorem

Suppose we write x to refer to the value of $\cos\theta$. Then we have proven that

$$\cos(2\theta) = 2x^2 - 1$$

All the Good Jokes Argon

Theorem

Suppose we write x to refer to the value of $\cos\theta$. Then we have proven that

$$\cos(2\theta) = 2x^2 - 1$$

To double check, suppose $x = \cos\theta = -0.2$. Then $\cos(2\theta) = 2x^2 - 1 =$

All the Good Jokes Argon

Theorem

Suppose we write x to refer to the value of $\cos\theta$. Then we have proven that

$$\cos(2\theta) = 2x^2 - 1$$

To double check, suppose $x = \cos\theta = -0.2$. Then $\cos(2\theta) = 2x^2 - 1 = -0.92$. *Bingo*

Fishiness Abounds

- Has anyone noticed anything suspicious?

Fishiness Abounds

- Has anyone noticed anything suspicious?
- What do you reckon will be the case if we take $x = \cos\theta$ and compute $4x^3 - 3x$?

Fishiness Abounds

- Has anyone noticed anything suspicious?
- What do you reckon will be the case if we take $x = \cos\theta$ and compute $4x^3 - 3x$?
- How can we visually confirm this?

Fishiness Abounds

- Has anyone noticed anything suspicious?
- What do you reckon will be the case if we take $x = \cos\theta$ and compute $4x^3 - 3x$?
- How can we visually confirm this?
- And what about $8x^4 - 8x^2 + 1$?

It All Makes Sense

Theorem

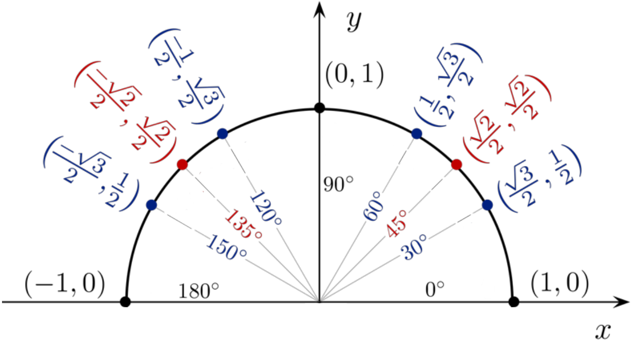
Substitute $x = \cos\theta$ *in the polynomial formula for the degree* n *neutron graph. Then the result will simplify to* $\cos(n\theta)$.

Let's check this out for $n = 5$. How does this result explain why the graph is so dense?

It All Makes Sense

Consider $16x^5 - 20x^3 + 5x$.

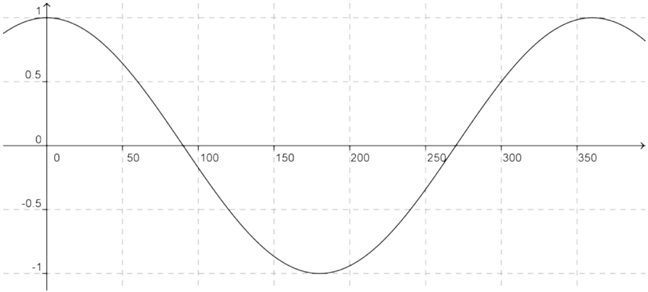
- $0° \leq \theta \leq 180° \implies x = 1$ to -1



It All Makes Cents

Consider $16x^5 - 20x^3 + 5x$.

- $0^\circ \leq \theta \leq 180^\circ \implies x = 1$ to -1
- Using $x = \cos\theta$ gives $y = \cos(5\theta)$



It All Makes Sense

Consider $16x^5 - 20x^3 + 5x$.

- $0° \leq \theta \leq 180° \implies x = 1$ to -1
- Using $x = \cos\theta$ gives $y = \cos(5\theta)$
- Therefore $-1 \leq y \leq 1$, and all of the maximums and minimums of the graph occur along the boundaries $y = 1$ and $y = -1$ of the unit window.

It All Makes Sense

Consider $16x^5 - 20x^3 + 5x$.

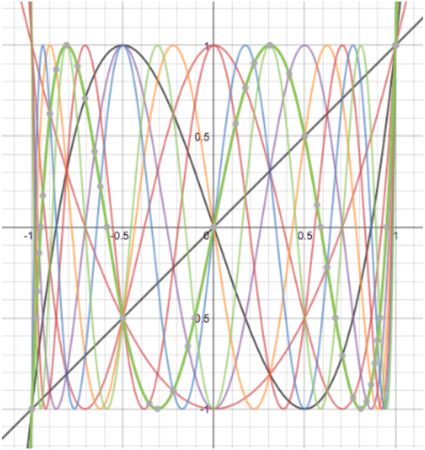
- $0° \leq \theta \leq 180° \implies x = 1$ to -1
- Using $x = \cos\theta$ gives $y = \cos(5\theta)$
- Therefore $-1 \leq y \leq 1$, and all of the maximums and minimums of the graph occur along the boundaries $y = 1$ and $y = -1$ of the unit window.

This even explains the overlapping graphs!

Once More With Feeling

Here are the neutron graphs up to degree 9.

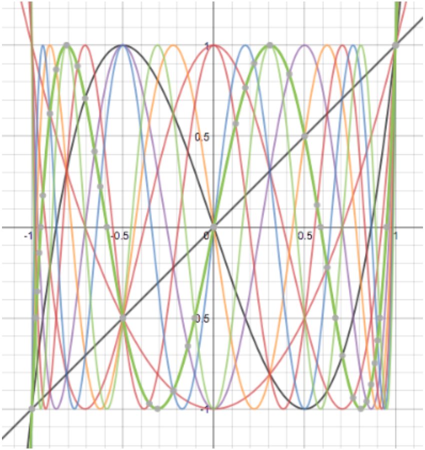
$$x = -\frac{1}{2}$$
$$x = \cos\theta$$
$$\theta =$$



Once More With Feeling

Here are the neutron graphs up to degree 9.

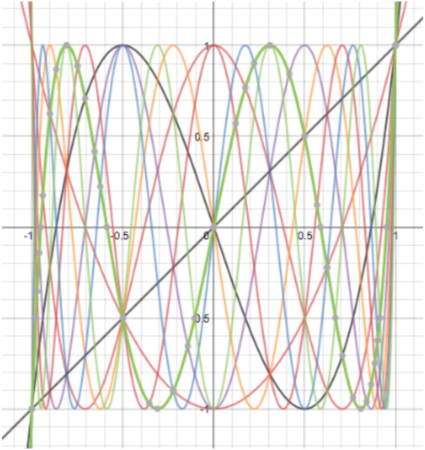
$$x = -\frac{1}{2}$$
$$x = \cos\theta$$
$$\theta = \mathbf{120}^\circ$$



Once More With Feeling

Here are the neutron graphs up to degree 9.

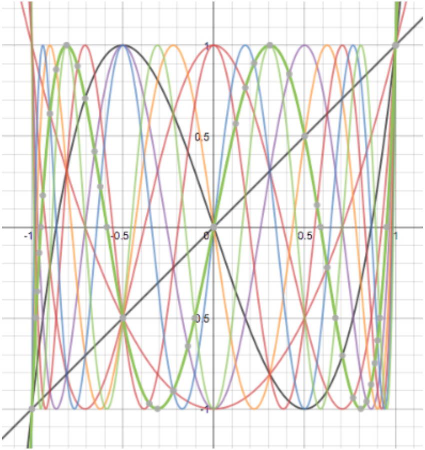
$$x = -\frac{1}{2}$$
$$x = \cos\theta$$
$$\theta = \mathbf{120}^\circ$$
$$2\theta = 240^\circ$$
$$3\theta = 360^\circ$$
$$4\theta = 480^\circ$$
$$5\theta = 600^\circ$$



Once More With Feeling

Here are the neutron graphs up to degree 9.

$$x = -\tfrac{1}{2}$$
$$x = \cos\theta$$
$$\theta = \mathbf{120^\circ}$$
$$2\theta = 240^\circ$$
$$3\theta = 360^\circ$$
$$4\theta = 480^\circ$$
$$5\theta = 600^\circ$$



y-values:

$$\cos(2\theta)$$
$$\cos(3\theta)$$
$$\cos(4\theta)$$
$$\cos(5\theta)$$

My Kingdom For a Proof

Our theorem explains everything. If only we could prove it. Let's assume that the result is true for $n = 2$, $n = 3$ and $n = 4$. Can we use this to get anywhere?

Here's the degree five neutron polynomial:

$$16x^5 - 20x^3 + 5x$$

My Kingdom For a Proof

Our theorem explains everything. If only we could prove it. Let's assume that the result is true for $n = 2$, $n = 3$ and $n = 4$. Can we use this to get anywhere?

Here's the degree five neutron polynomial:

$$2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x)$$

My Kingdom For a Poof

Our theorem explains everything. If only we could prove it. Let's assume that the result is true for $n = 2$, $n = 3$ and $n = 4$. Can we use this to get anywhere?

Here's the degree five neutron polynomial:

$$2\cos\theta(8\cos^4\theta - 8\cos^2\theta + 1) - (4\cos^3\theta - 3\cos\theta)$$

My Phylum For a Proof

Our theorem explains everything. If only we could prove it. Let's assume that the result is true for $n = 2$, $n = 3$ and $n = 4$. Can we use this to get anywhere?

Here's the degree five neutron polynomial:

$$2\cos\theta(8\cos^4\theta - 8\cos^2\theta + 1) - (\cos 3\theta)$$

My Kingdom For a Proof

Our theorem explains everything. If only we could prove it. Let's assume that the result is true for $n = 2$, $n = 3$ and $n = 4$. Can we use this to get anywhere?

Here's the degree five neutron polynomial:

$$(2\cos\theta\cos 4\theta) - \cos 3\theta$$

I Used To Like Trig

Our theorem explains everything. If only we could prove it. Let's assume that the result is true for $n = 2$, $n = 3$ and $n = 4$. Can we use this to get anywhere?

Here's the degree five neutron polynomial:

$$(\cos 5\theta + \cos 3\theta) - \cos 3\theta$$

Victory = Lunch

Our theorem explains everything. If only we could prove it. Let's assume that the result is true for $n = 2$, $n = 3$ and $n = 4$. Can we use this to get anywhere?

Here's the degree five neutron polynomial:

$$\cos 5\theta.$$

And you're done!

See You Tomorrow

Trous tamers, don't forget to write up your proof to turn in this afternoon!

See You Tomorrow

Dr.V will be in the main lounge from 1:00 to 1:30 today to help out and shoot pool.

There Is Nothing To See