≡ Prickly Primes ≡

MathπAth 2018 • Lewis & Clark College

Sam Vandervelde • Proof School • July 3, 2018

Prime Personalities

There are four main categories of **primes**:

Prime Personalities

There are four main categories of **primes**:

- Primes ending in 9 (**fickle primes**)

Prime Personalities

There are four main categories of **primes**:

- Primes ending in 9 (**fickle primes**)
- Primes ending in 7 (**sickly primes**)

Prime Personalities

There are four main categories of **primes**:

- Primes ending in 9 (**fickle primes**)
- Primes ending in 7 (**sickly primes**)
- Primes ending in 3 (**tricky primes**)

Prime Personalities

There are four main categories of **primes**:

- Primes ending in 9 (**fickle primes**)
- Primes ending in 7 (**sickly primes**)
- Primes ending in 3 (**tricky primes**)
- Primes ending in 1 (**prickly primes**)

Prime Personalities

There are four main categories of **primes**:

- Primes ending in 9 (**fickle primes**)
- Primes ending in 7 (**sickly primes**)
- Primes ending in 3 (**tricky primes**)
- Primes ending in 1 (**prickly primes**)
- All the rest of them:

Prime Personalities

There are four main categories of **primes**:

- Primes ending in 9 (**fickle primes**)
- Primes ending in 7 (**sickly primes**)
- Primes ending in 3 (**tricky primes**)
- Primes ending in 1 (**prickly primes**)
- All the rest of them: **2 and 5**

Prickly Trivia

What do you know about **prickly primes**?

Prickly Trivia

What do you know about **prickly primes**?

The first prickly prime is: **1**

Prickly Trivia

What do you know about **prickly primes**?

The first prickly prime is: **11**

The seventh prickly prime is:

Prickly Trivia

What do you know about **prickly primes**?

The first prickly prime is: **11**

The seventh prickly prime is: **131**

How many prickly primes are there?

Prickly Trivia

What do you know about **prickly primes**?

The first prickly prime is: **11**

The seventh prickly prime is: **131**

How many prickly primes are there? ∞

How hard is this to prove?

Prickly Trivia

What do you know about **prickly primes**?

The first prickly prime is: **11**

The seventh prickly prime is: **131**

How many prickly primes are there? ∞

How hard is this to prove? **KindaHard**$^{\text{TM}}$

Prickly Trivia

What fraction of primes are prickly?

Prickly Trivia

What fraction of primes are prickly? **1/4**

How hard is this to prove?

Prickly Trivia

What fraction of primes are prickly? **1/4**

How hard is this to prove? **SoopaHard**™

What type of cactus is this?

Prickly Trivia

What fraction of primes are prickly? **1/4**

How hard is this to prove? **SoopaHard**^{TM}

What type of cactus is this? **prickly pear**

Baby Steps

Our goal will be to prove that there are infinitely many prickly primes. Recall that this is KindaHard^{TM}. We'd better go easy.

Baby Steps

Our goal will be to prove that there are infinitely many prickly primes. Recall that this is KindaHard^{TM}. We'd better go easy.

How many primes are there?

Baby Steps

Our goal will be to prove that there are infinitely many prickly primes. Recall that this is KindaHard^{TM}. We'd better go easy.

How many primes are there? ∞

Who is credited with the first proof?

Baby Steps

Our goal will be to prove that there are infinitely many prickly primes. Recall that this is KindaHard^{TM}. We'd better go easy.

How many primes are there? ∞

Who is credited with the first proof? **Euclid**

We'll employ the Doritos proof strategy.

Tasty Proof Strategy

Crunch All You Like

Here are all the primes:

$$2, 3, 5, 7, 11, 13, \ldots, 15\,485\,863.$$

Crunch All You Like

Here are the first million primes:

$$2, 3, 5, 7, 11, 13, \ldots, 15\,485\,863.$$

Convince your neighbor that there is some other prime out there that does not already appear on this list, by building it.

Crunch All You Like

Theorem (Euclid)

There are infinitely many primes.

Crunch All You Like

Here are a lot of prickly primes:

$$11, 31, 41, 61, \ldots, 67\,868\,011.$$

Convince your neighbor that there is some other prickly prime out there that does not already appear on this list, by building it.

That Didn't Go So Well

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

$$\text{Let } \boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$$

- $P + 1 =$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

$$\text{Let } \boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$$

- $P + 1 = 2^5 \cdot 3^5 \cdot 13 \cdot 599$
- $10P + 1 =$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

- $P + 1 = 2^5 \cdot 3^5 \cdot 13 \cdot 599$
- $10P + 1 = 3^2 \cdot 4657 \cdot 14447$
- $P + 2 =$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

- $P + 1 = 2^5 \cdot 3^5 \cdot 13 \cdot 599$
- $10P + 1 = 3^2 \cdot 4657 \cdot 14447$
- $P + 2 = 1303 \cdot 46471$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

- $P + 1 = 2^5 \cdot 3^5 \cdot 13 \cdot 599$
- $10P + 1 = 3^2 \cdot 4657 \cdot 14447$
- $P + 2 = 1303 \cdot 46471$ **but**
- $11 \cdot 31 \cdot 41 \cdot 61 + 2 = 3 \cdot 67 \cdot 4243$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

$$\text{Let } \boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$$

- $P^2 + 1 =$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

- $P^2 + 1 = 2 \cdot 2957 \cdot 619971204773$
- $10P^2 + 1 =$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

- $P^2 + 1 = 2 \cdot 2957 \cdot 619971204773$
- $10P^2 + 1 = 7 \cdot 52378710071821 73$
- $P^2 + 2 =$

That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

- $P^2 + 1 = 2 \cdot 2957 \cdot 619971204773$
- $10P^2 + 1 = 7 \cdot 5237871007182173$
- $P^2 + 2 = 3 \cdot 163 \cdot 7497974856907$
- $P^3 + 17 =$

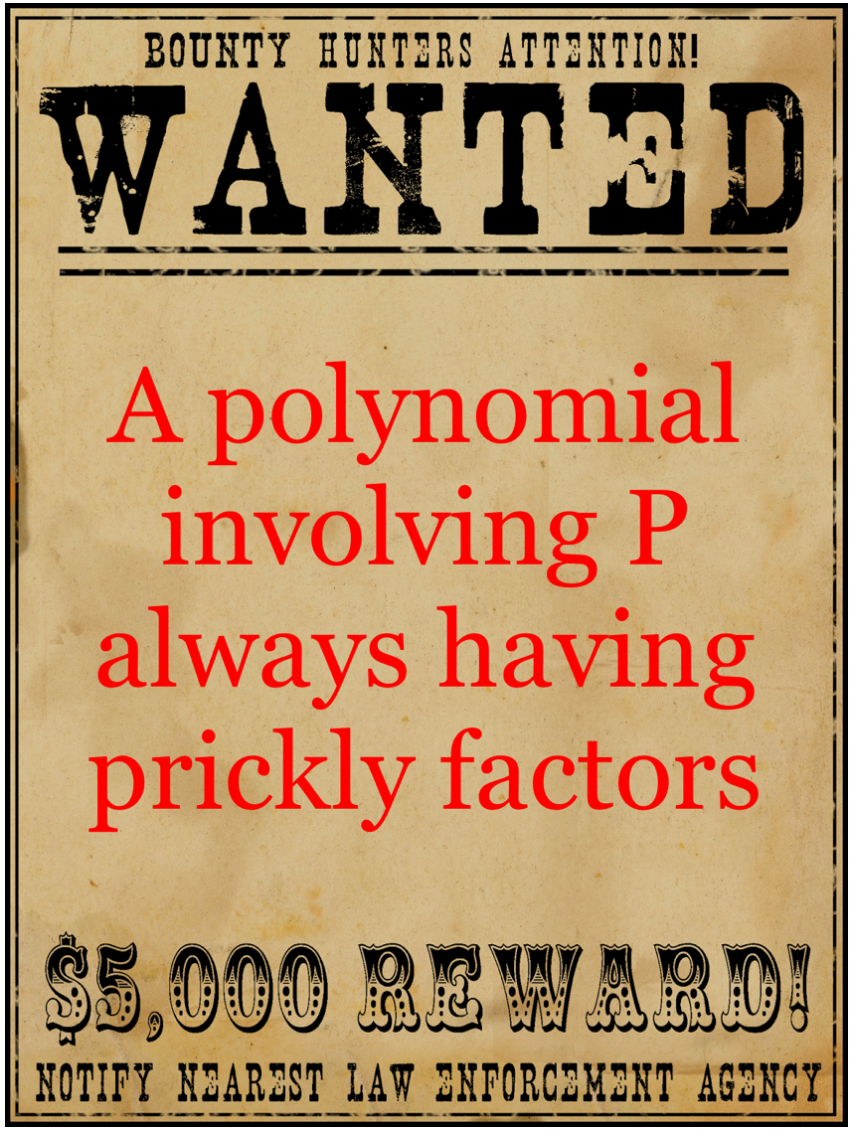
That Didn't Go So Well

Let's try to build a new prickly prime using the first five of them: 11, 31, 41, 61, 71.

Let $\boxed{P = 11 \cdot 31 \cdot 41 \cdot 61 \cdot 71.}$

- $P^2 + 1 = 2 \cdot 2957 \cdot 619971204773$
- $10P^2 + 1 = 7 \cdot 5237871007182173$
- $P^2 + 2 = 3 \cdot 163 \cdot 7497974856907$
- $P^3 + 17 =$ **forget it**

That Didn't Go So Well



Check it out

n	$n^2 + n + 1$	prime factors
1	3	3
2	7	7
3	13	13
4	21	3, 7
5	31	31
6	43	43
7	57	3, 19
8	73	73
9	91	7, 13

What is going on with the primes?

Check it out

Here are all the prime factors that occur for numbers of the form $n^2 + n + 1$.

$$3,\ 7,\ 13,\ 19,\ 31,\ 37,\ 43,\ 61,\ 67,\ 73, \ldots$$

What do all these primes have in common?

Check it out

Here are all the prime factors that occur for numbers of the form $n^2 + n + 1$.

$$3,\ 7,\ 13,\ 19,\ 31,\ 37,\ 43,\ 61,\ 67,\ 73, \ldots$$

What do all these primes have in common?

TIP: it's not about the final digit; rather, it has to do with 3, the first prime in the list.

Check it out

I'm Convinced

Based on the data we conjecture that for any positive integer n, the only prime factors of

$$n^2 + n + 1$$

are primes of the form $3k + 1$, like

$$7,\ 13,\ 19,\ 31,\ 37,\ 43,\ \ldots$$

(And possibly a factor of 3.)

I'm Convinced

Based on the data we conjecture that for any positive integer n, the only prime factors of

$$n^2 + n + 1$$

are primes of the form $3k + 1$, like

$$7,\ 13,\ 19,\ 31,\ 37,\ 43,\ \ldots$$

(And possibly a factor of 3.) **That's nice.**

Prickly Review

What type of cactus is pictured?

Prickly Review

What type of cactus is pictured? **Saguaro**

So What?

What good is knowing about the prime factors of $n^2 + n + 1$?

So What?

What good is knowing about the prime factors of $n^2 + n + 1$? Let's call such primes, of the form $3k + 1$, *principled primes*.

So What?

What good is knowing about the prime factors of $n^2 + n + 1$? Let's call such primes, of the form $3k + 1$, *principled primes*.

Theorem

There are infinitely many principled primes.

So What?

What good is knowing about the prime factors of $n^2 + n + 1$? Let's call such primes, of the form $3k + 1$, *principled primes*.

Theorem

There are infinitely many principled primes.

Remember our proof strategy!

Crunch All You Like

Here are a lot of principled primes:

$$7, 13, 19, 31, 37 \ldots, 15\,485\,863.$$

Convince your neighbor that there is some other principled prime out there that does not already appear on this list, by building it.

Crunch All You Like

Here's an outline of the argument:

Crunch All You Like

Here's an outline of the argument:

- Let $P = (7)(13)(19)\cdots(15\,485\,863)$.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (7)(13)(19)\cdots(15\,485\,863)$.
- Compute the positive integer $P^2 + P + 1$.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (7)(13)(19)\cdots(15\,485\,863)$.
- Compute the positive integer $P^2 + P + 1$.
- Argue this value is not divisible by any principled prime already on the list.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (7)(13)(19)\cdots(15\,485\,863)$.
- Compute the positive integer $P^2 + P + 1$.
- Argue this value is not divisible by any principled prime already on the list.
- Deal with the (single) factor of 3.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (7)(13)(19)\cdots(15\,485\,863)$.
- Compute the positive integer $P^2 + P + 1$.
- Argue this value is not divisible by any principled prime already on the list.
- Deal with the (single) factor of 3.
- Deduce a new principled prime exists.

Prickly Review

What type of cactus is pictured?

Prickly Review

What type of cactus is pictured? **Golden barrel**

Go To The Principle

What is so special about $n^2 + n + 1$ that it has only principle prime factors, and why would this polynomial have anything to do with the number 3?

Go To The Principle

What is so special about $n^2 + n + 1$ that it has only principle prime factors, and why would this polynomial have anything to do with the number 3?

It appears in the factorization

$$n^3 - 1 = (n - 1)(n^2 + n + 1).$$

What could this all mean?

Go To The Principle

Theorem

Every prime factor p of $n^2 + n + 1$, other than 3, must be principled, i.e. must be of the form $p = 3k + 1$.

Go To The Principle

Theorem

Every prime factor p of $n^2 + n + 1$, other than 3, must be principled, i.e. must be of the form $p = 3k + 1$.

- Given n, suppose $47 \mid (n^2 + n + 1)$.

Go To The Principle

Theorem

Every prime factor p of $n^2 + n + 1$, other than 3, must be principled, i.e. must be of the form $p = 3k + 1$.

- Given n, suppose $47 \mid (n^2 + n + 1)$.
- Therefore $47 \mid (n^3 - 1)$.

Go To The Principle

Theorem

Every prime factor p of $n^2 + n + 1$, other than 3, must be principled, i.e. must be of the form $p = 3k + 1$.

- Given n, suppose $47 \mid (n^2 + n + 1)$.
- Therefore $47 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 47$.

Go To The Principle

- Given n, suppose $47 \mid (n^2 + n + 1)$.
- Therefore $47 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 47$.

Go To The Principle

- Given n, suppose $47 \mid (n^2 + n + 1)$.
- Therefore $47 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 47$.
- We know $n^{46} \equiv 1 \bmod 47$.

Go To The Principle

- Given n, suppose $47 \mid (n^2 + n + 1)$.
- Therefore $47 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 47$.
- We know $n^{46} \equiv 1 \bmod 47$.
- We deduce that $n \equiv 1 \bmod 47$.

Go To The Principle

- Given n, suppose $47 \mid (n^2 + n + 1)$.
- Therefore $47 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 47$.
- We know $n^{46} \equiv 1 \bmod 47$.
- We deduce that $n \equiv 1 \bmod 47$.
- But this gives $n^2 + n + 1 \equiv 3 \bmod 47$,
- which contradicts $47 \mid (n^2 + n + 1)$.

Go To The Principle

Why doesn't this eliminate all prime factors?

Go To The Principle

Why doesn't this eliminate all prime factors?

- Given n, suppose $37 \mid (n^2 + n + 1)$.

Go To The Principle

Why doesn't this eliminate all prime factors?

- Given n, suppose $37 \mid (n^2 + n + 1)$.
- Therefore $37 \mid (n^3 - 1)$.

Go To The Principle

Why doesn't this eliminate all prime factors?

- Given n, suppose $37 \mid (n^2 + n + 1)$.
- Therefore $37 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 37$.

Go To The Principle

Why doesn't this eliminate all prime factors?

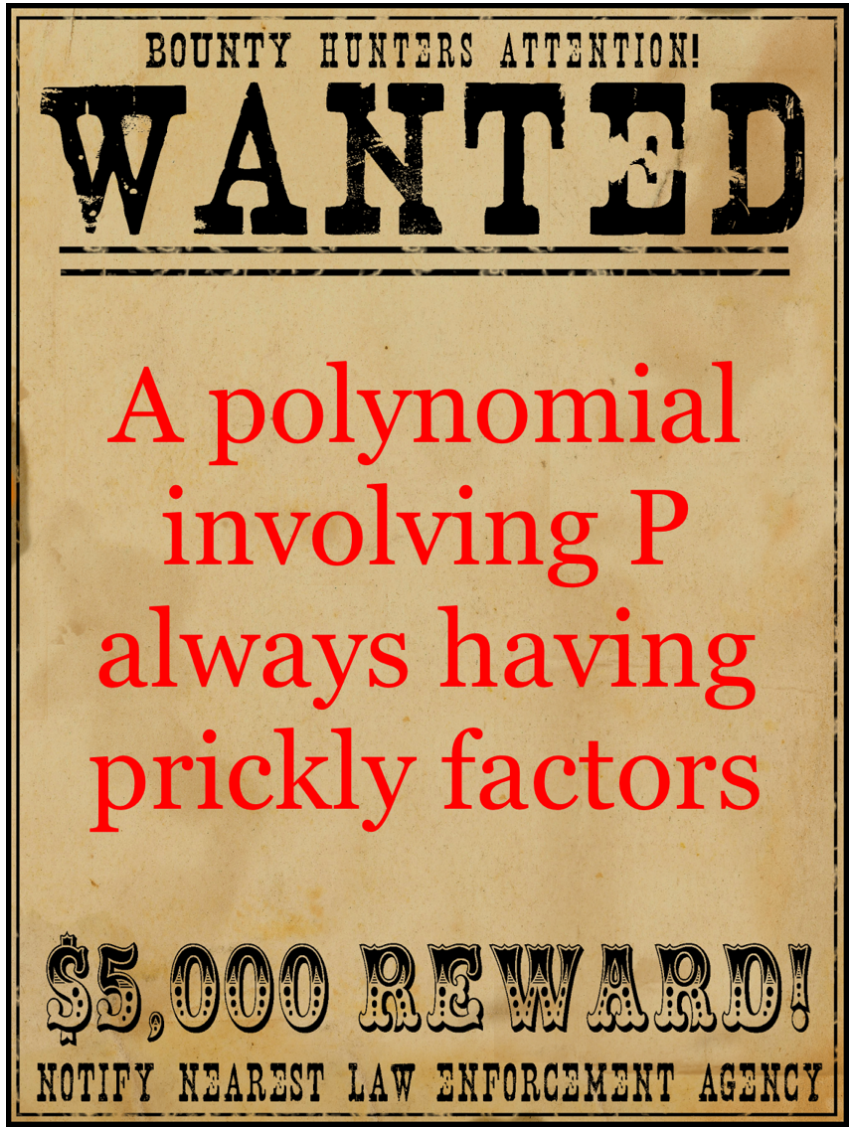
- Given n, suppose $37 \mid (n^2 + n + 1)$.
- Therefore $37 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 37$.
- We know $n^{36} \equiv 1 \bmod 37$.

Go To The Principle

Why doesn't this eliminate all prime factors?

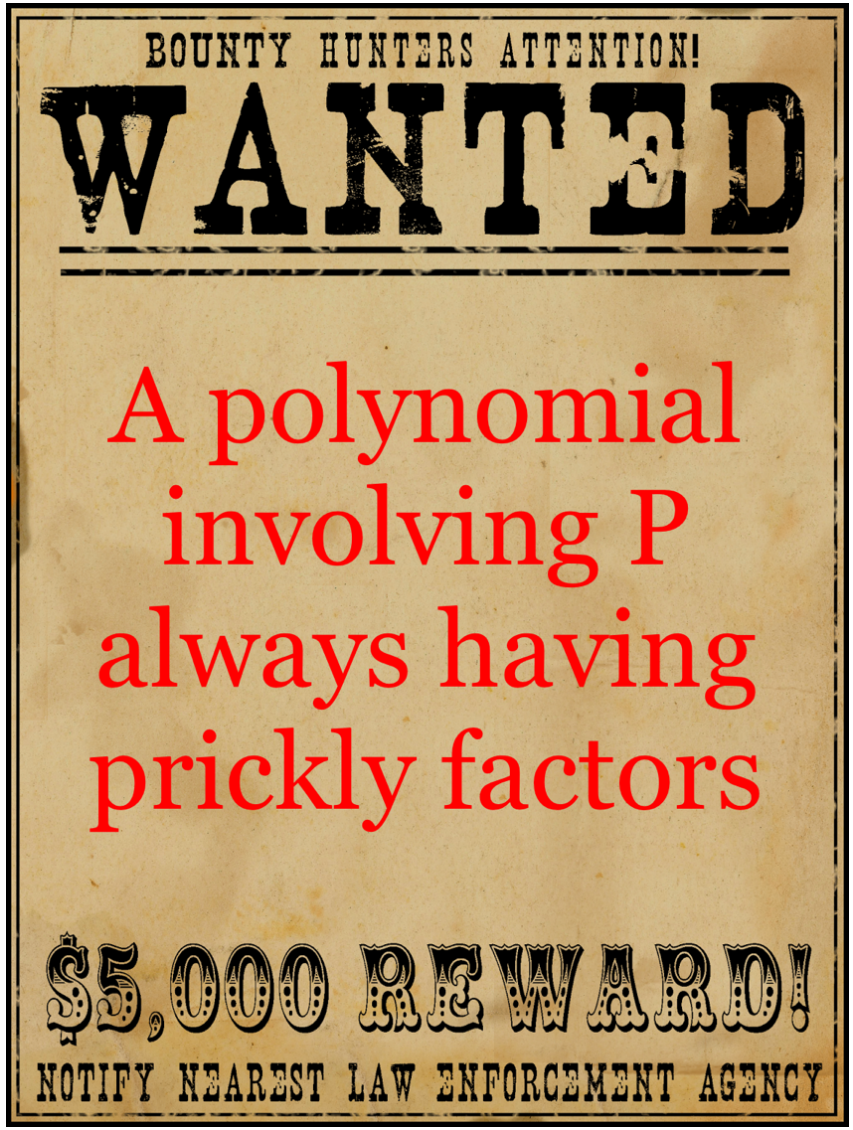
- Given n, suppose $37 \mid (n^2 + n + 1)$.
- Therefore $37 \mid (n^3 - 1)$.
- This means that $n^3 \equiv 1 \bmod 37$.
- We know $n^{36} \equiv 1 \bmod 37$.
- We deduce that $1 \equiv 1 \bmod 37$,
- *and nothing breaks!*

Scoring Five Grand



Scoring Five Grand

Where shall

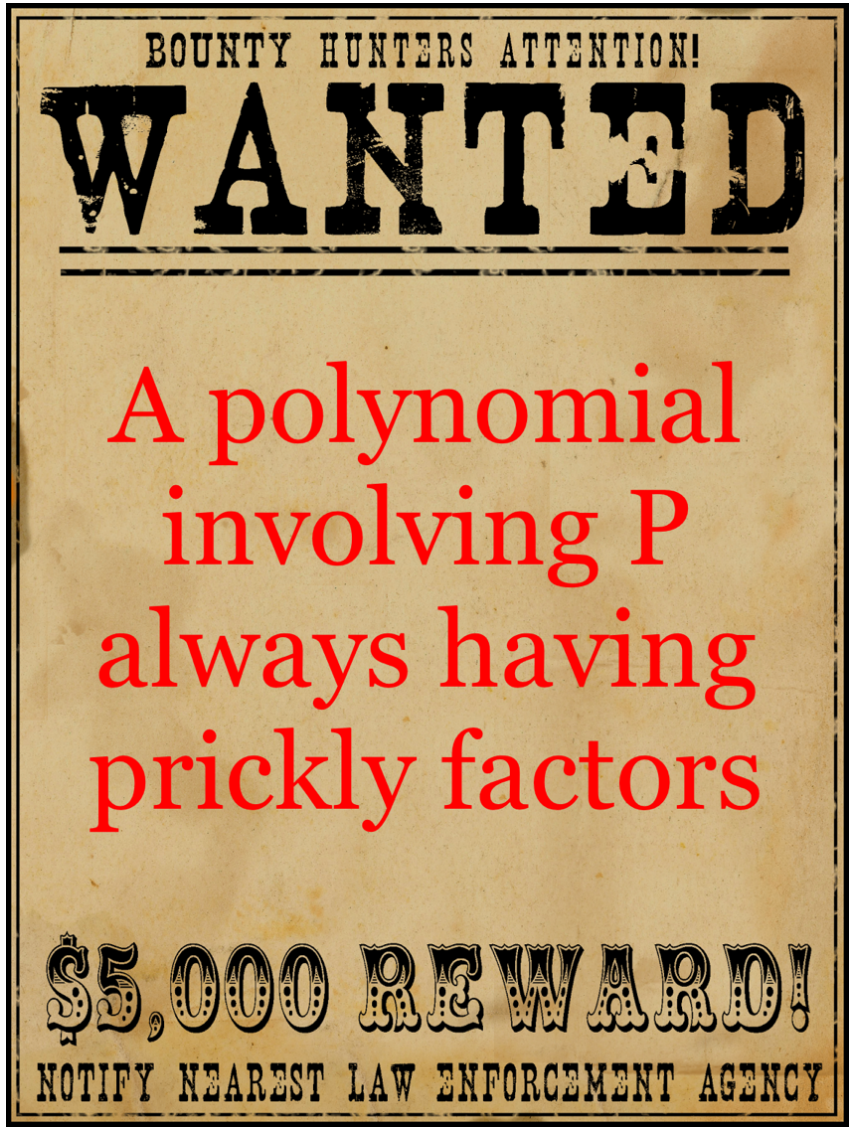


we look for it?

Scoring Five Grand

Where shall

GOOD



we look for it?

IDEA!

Scoring Five Grand

Who can get us started?

$$n^{10} - 1 \; =$$

Scoring Five Grand

Who can get us started?

$$n^{10} - 1 = (n^5 - 1)(n^5 + 1)$$

Scoring Five Grand

Who can get us started?

$$n^{10} - 1 = (n^5 - 1)(n^5 + 1)$$

$$(n^5 - 1) =$$

Scoring Five Grand

Who can get us started?

$$\begin{aligned} n^{10} - 1 &= (n^5 - 1)(n^5 + 1) \\ (n^5 - 1) &= (n - 1)(n^4 + n^3 + n^2 + n + 1) \\ (n^5 + 1) &= \end{aligned}$$

Scoring Five Grand

Who can get us started?

$$\begin{aligned} n^{10} - 1 &= (n^5 - 1)(n^5 + 1) \\ (n^5 - 1) &= (n - 1)(n^4 + n^3 + n^2 + n + 1) \\ (n^5 + 1) &= (n + 1)(n^4 - n^3 + n^2 - n + 1) \end{aligned}$$

Which of those quartics shall we use?

Scoring Five Grand

Who can get us started?

$$\begin{aligned} n^{10} - 1 &= (n^5 - 1)(n^5 + 1) \\ (n^5 - 1) &= (n - 1)(n^4 + n^3 + n^2 + n + 1) \\ (n^5 + 1) &= (n + 1)(n^4 - n^3 + n^2 - n + 1) \end{aligned}$$

Which of those quartics shall we use?

Either one will do. Let's give it a try!

Prickly Review

What type of cactus is pictured?

Prickly Review

What type of cactus is pictured? **San Pedro**

Scoring Five Grand

n	$n^4 - n^3 + n^2 - n + 1$	prime factors
12	19141	19141
13	26521	11, 241
14	35855	5, 71, 101
15	47461	31, 1531
16	61681	61681
17	78881	11, 71, 101
18	99451	11, 9041

Scoring Five Grand

Let's try out a SICKLY prime.

- Suppose $47 \mid (n^4 - n^3 + n^2 - n + 1)$.

Scoring Five Grand

Let's try out a SICKLY prime.

- Suppose $47 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $47 \mid (n^5 + 1)$.

Scoring Five Grand

Let's try out a SICKLY prime.

- Suppose $47 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $47 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 47$.

Scoring Five Grand

Let's try out a SICKLY prime.

- Suppose $47 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $47 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 47$.
- We know $n^{46} \equiv 1 \bmod 47$.

Scoring Five Grand

Let's try out a SICKLY prime.

- Suppose $47 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $47 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 47$.
- We know $n^{46} \equiv 1 \bmod 47$.
- We deduce that $n \equiv -1 \bmod 47$.

Scoring Five Grand

Let's try out a SICKLY prime.

- Suppose $47 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $47 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 47$.
- We know $n^{46} \equiv 1 \bmod 47$.
- We deduce that $n \equiv -1 \bmod 47$.
- Hence $n^4 - n^3 + n^2 - n + 1 \equiv 5 \bmod 47$,

Scoring Five Grand

Let's try out a SICKLY prime.

- Suppose $47 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $47 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 47$.
- We know $n^{46} \equiv 1 \bmod 47$.
- We deduce that $n \equiv -1 \bmod 47$.
- Hence $n^4 - n^3 + n^2 - n + 1 \equiv 5 \bmod 47$,
- Contrary to $47 \mid (n^4 - n^3 + n^2 - n + 1)$.

Scoring Five Grand

No doubt TRICKY and FICKLE primes work out the same way, and we're done, right?

Scoring Five Grand

No doubt TRICKY and FICKLE primes work out the same way, and we're done, right?

Aw, cmon, don't you trust me?

Scoring Five Grand

No doubt TRICKY and FICKLE primes work out the same way, and we're done, right?

Aw, cmon, don't you trust me?

Fine, we'll do one more case together, then you can confirm the last one on your own. Which type of prime shall we tackle next?

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv -1 \bmod 83$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv -1 \bmod 83$.
- Hence $n^4 - n^3 + n^2 - n + 1 \equiv 5 \bmod 83$,

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv -1 \bmod 83$.
- Hence $n^4 - n^3 + n^2 - n + 1 \equiv 5 \bmod 83$,
- Contrary to $83 \mid (n^4 - n^3 + n^2 - n + 1)$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv -1 \bmod 83$.
- Hence $n^4 - n^3 + n^2 - n + 1 \equiv 5 \bmod 83$,

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv -1 \bmod 83$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n^2 \equiv 1 \bmod 83$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv \pm 1 \bmod 83$.

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv \pm 1 \bmod 83$.
- So $n^4 - n^3 + n^2 - n + 1 \equiv 1, 5 \bmod 83$,

Scoring Five Grand

Great, here's a TRICKY prime.

- Suppose $83 \mid (n^4 - n^3 + n^2 - n + 1)$.
- Therefore $83 \mid (n^5 + 1)$.
- This means that $n^5 \equiv -1 \bmod 83$.
- We know $n^{82} \equiv 1 \bmod 83$.
- We deduce that $n \equiv \pm 1 \bmod 83$.
- So $n^4 - n^3 + n^2 - n + 1 \equiv 1, 5 \bmod 83$,
- Contrary to $83 \mid (n^4 - n^3 + n^2 - n + 1)$.

We're Home Free Now

You should now be able to dispatch the FICKLE case with ease.

We're Home Free Now

You should now be able to dispatch the FICKLE case with ease.

Nothing can stop us now in our quest to demonstrate that there are infinitely many prickly primes.

Prickly Review

What type of cactus is pictured?

Prickly Review

What type of cactus is pictured? **Barbary Fig**

Prickle or Bust

Theorem

There are infinitely many prickly primes.

Remember our proof strategy!

Prickle or Bust

Theorem

There are infinitely many prickly primes.

Remember our proof strategy!

Here are a lot of prickly primes:

$$11, 31, 41, 61, \ldots, 67\,868\,011.$$

Let's prove that there exists a prickly prime that does not already appear on this list.

Crunch All You Like

Here's an outline of the argument:

Crunch All You Like

Here's an outline of the argument:

- Let $P = (5)(11)(31)\cdots(67\,868\,011)$.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (5)(11)(31)\cdots(67\,868\,011)$.
- Compute $P^4 - P^3 + P^2 - P + 1$.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (5)(11)(31)\cdots(67\,868\,011)$.
- Compute $P^4 - P^3 + P^2 - P + 1$.
- Argue this value is not divisible by any prickly prime already on the list.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (5)(11)(31)\cdots(67\,868\,011)$.
- Compute $P^4 - P^3 + P^2 - P + 1$.
- Argue this value is not divisible by any prickly prime already on the list.
- Finess the factors of 2 and 5

Crunch All You Like

Here's an outline of the argument:

- Let $P = (5)(11)(31)\cdots(67\,868\,011)$.
- Compute $P^4 - P^3 + P^2 - P + 1$.
- Argue this value is not divisible by any prickly prime already on the list.
- Finess the factors of 2 and 5
- Deduce a new prickly prime exists.

Crunch All You Like

Here's an outline of the argument:

- Let $P = (5)(11)(31)\cdots(67\,868\,011)$.
- Compute $P^4 - P^3 + P^2 - P + 1$.
- Argue this value is not divisible by any prickly prime already on the list.
- Finess the factors of 2 and 5
- Deduce a new prickly prime exists.
- *And You're Done!*

See You Tomorrow

Thanks for being such a great audience!