Proportions and layouts

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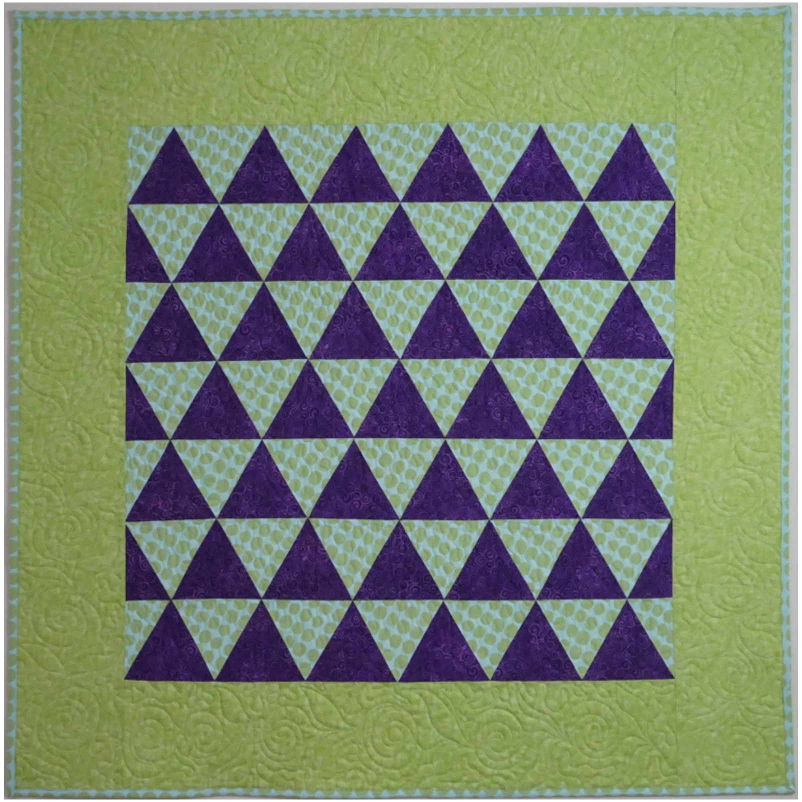
MathPath, Lewis & Clark College, Portland, Oregon, 25 July 2018

Proportions in a quilt



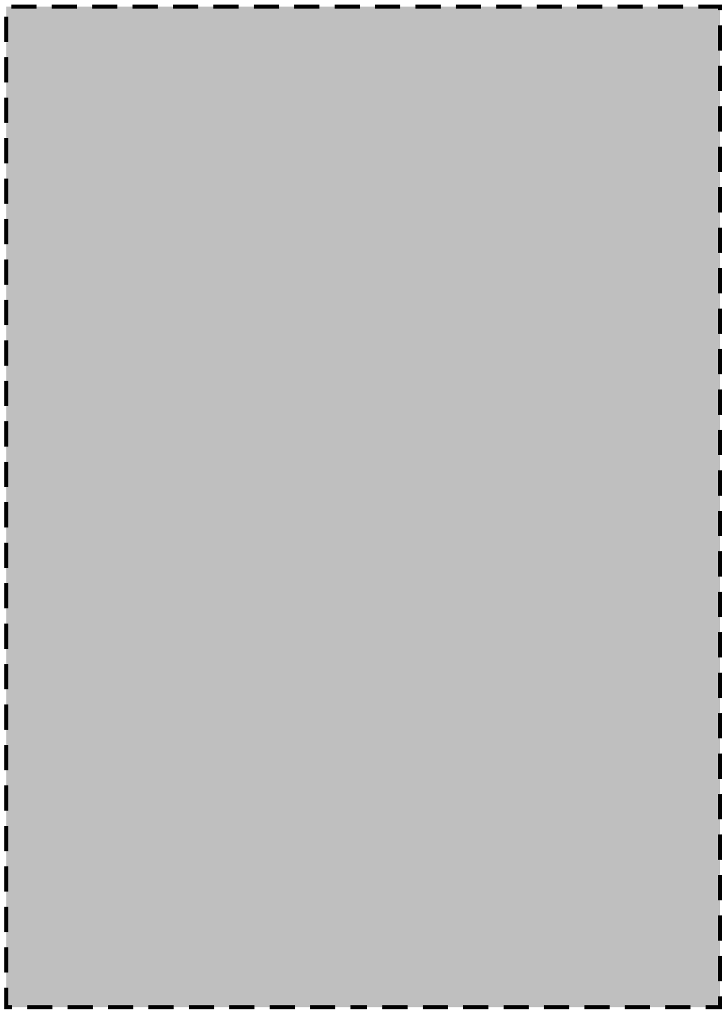
A quilt with n rows each with m bases of equilateral triangles has the ratio of height divided by width equal to:

$$\frac{n\frac{\sqrt{3}}{2}}{m} = \frac{n}{m}\frac{\sqrt{3}}{2}.$$

The proportion is close to R exactly when $\frac{m}{n} \approx \frac{\sqrt{3}}{2R}$.

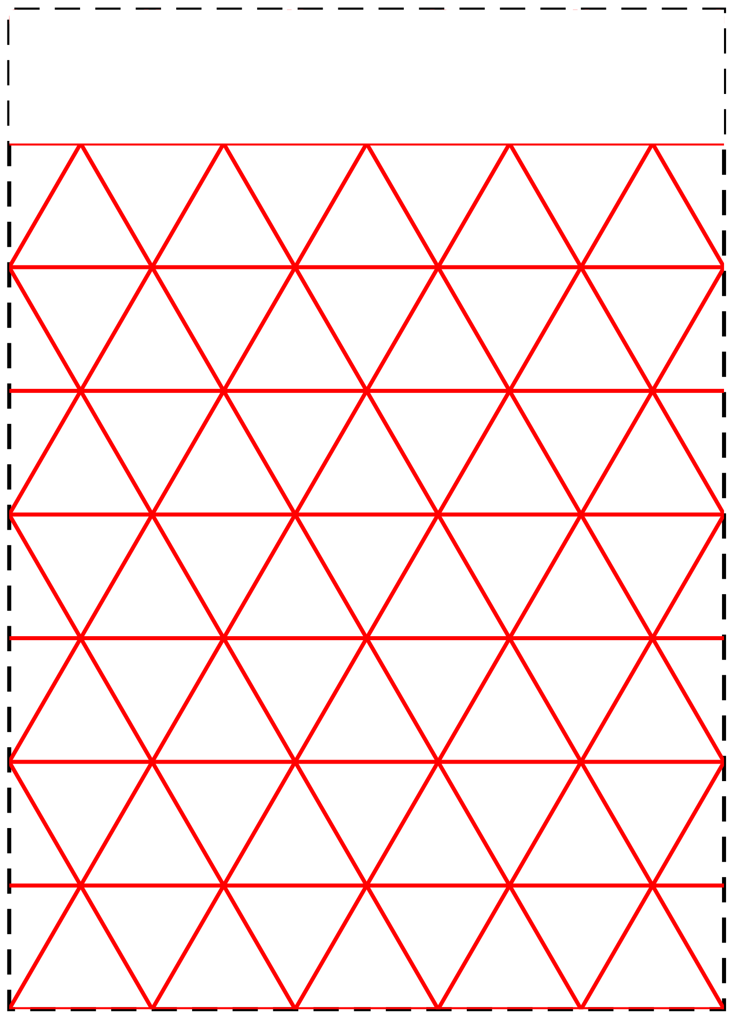
Since $\frac{\sqrt{3}}{2} \approx 0.866$, a first answer when $R = 1$ is $m = 866$ and $n = 1000$. A **better answer for a quilter** is $m = 433$ and $n = 500$.

We can get better rational approximations with continued fractions.



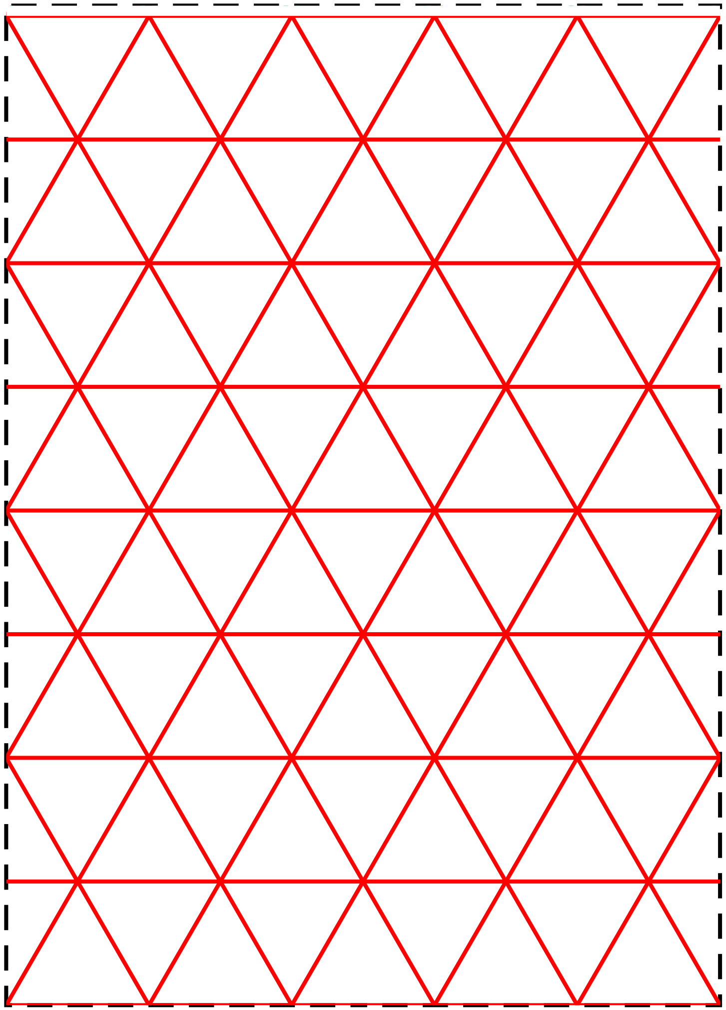
Fill with rows of equilateral triangles:

base = 5, height = 7



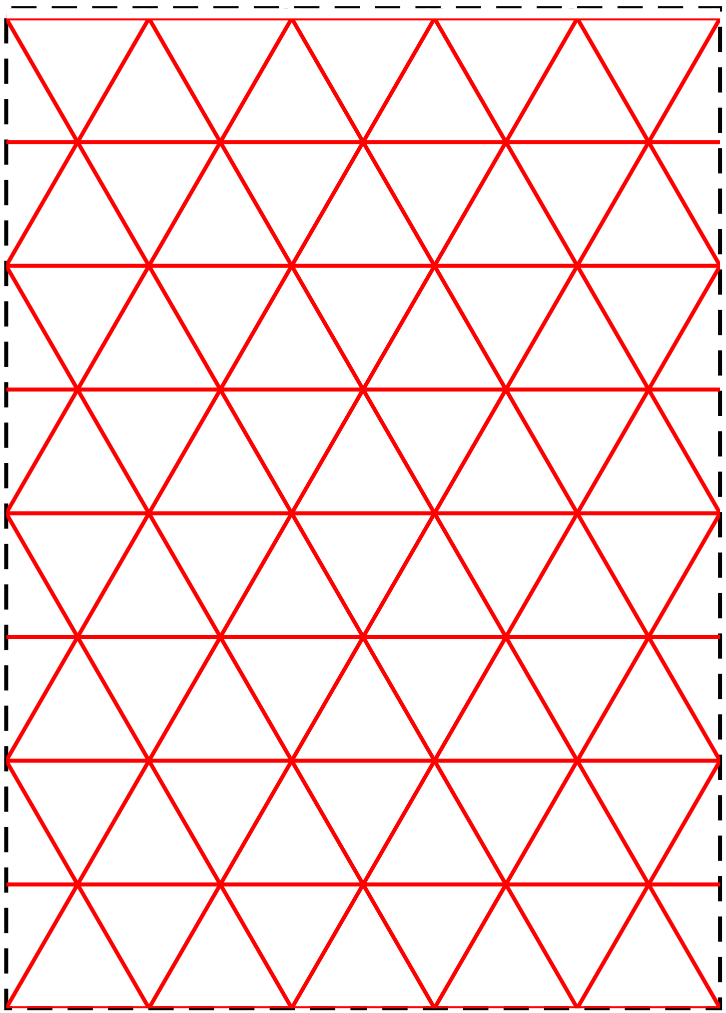
Fill with rows of equilateral triangles:

base = 5, height = 7
number of rows q = 7
number of columns p = 5



Fill with rows of equilateral triangles:

base = 5, height = 7
number of rows q = 8
number of columns p = 5



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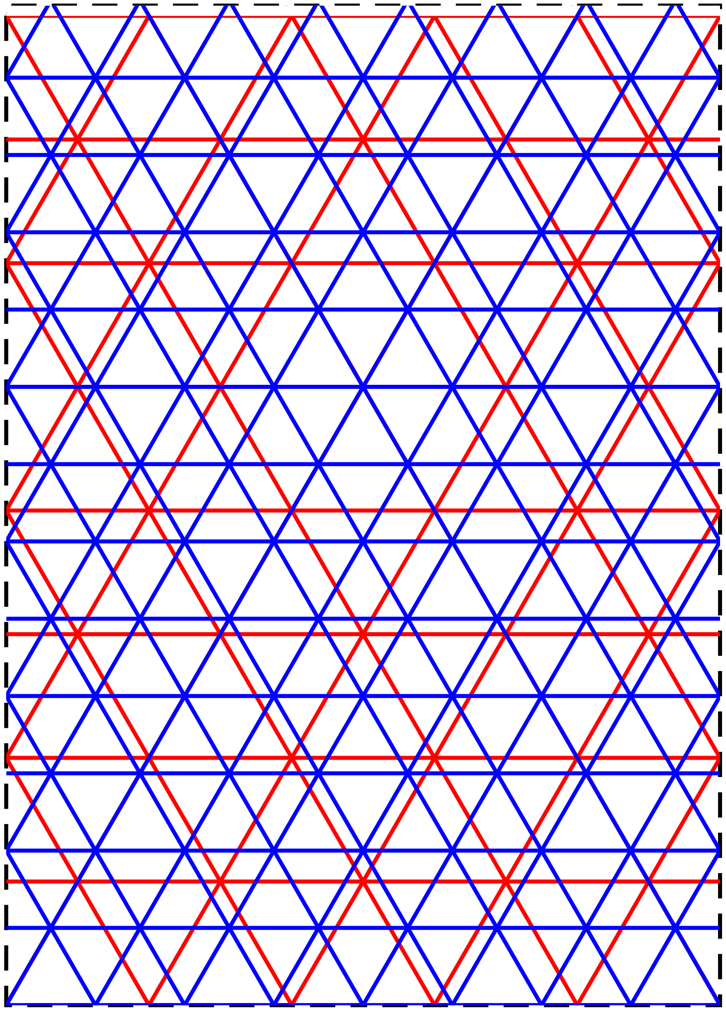
Can we do better?



Fill with rows of equilateral triangles:

base = 5, height = 7
red: number of rows q = 8
number of columns p = 5

blue: number of rows q = 13
number of columns p = 8

Can we do better?
We want p, q with $\frac{p}{q\frac{\sqrt{3}}{2}} \approx \frac{5}{7}$, i.e., we want

$$\frac{p}{q} \approx \frac{5}{7} \cdot \frac{\sqrt{3}}{2}.$$

Continued fractions

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So

$$r = a_0 + \frac{1}{r_1}.$$

Repeat the process from the previous page with r_1 in place of r. Stop with r_1 if r_1 is an integer, otherwise

$$a_1 = \lfloor r_1 \rfloor, \qquad \text{and} \qquad r_1 = a_1 + \frac{1}{r_2}$$

for some positive number $r_2 > 1$.

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Assuming that none of the r_i are integers, we continue:

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This is a **continued fraction expansion of** r.

Notation for continued fractions:

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}} = [a_0; a_1, a_2, a_3, a_4, \ldots].$$

$$\boxed{\textbf{Continued fractions on a special case}}$$

$$\frac{\sqrt{3}}{2} = 0 + \cfrac{1}{\frac{2}{\sqrt{3}}}$$

$\boxed{\textbf{Continued fractions}}$

$$\begin{aligned}\frac{\sqrt{3}}{2} &= 0 + \frac{1}{\frac{2}{\sqrt{3}}} \\ &= 0 + \frac{1}{1 + \left(\frac{2}{\sqrt{3}} - 1\right)}\end{aligned}$$

$$\left(\frac{2}{\sqrt{3}} \approx 1.1547\right)$$

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$$= 0 + \frac{1}{1 + \frac{1}{6 + ??}}$$

Continued fractions

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$$= 0 + \cfrac{1}{1 + \frac{1}{6 + \frac{1}{2 + \frac{1}{6 + \frac{1}{2 + \cdots}}}}}$$

What to do with continued fractions

The **convergents** of $x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cdots}}}}}$ are:

$$a_0,\ \ a_0 + \frac{1}{a_1},\ \ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}},\ \ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}},\ \ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}},\ \ \ldots .$$

The **convergents** of $\frac{\sqrt{3}}{2} = 0 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{2 + \cdots}}}}}$ are:

$$0,\ \ \frac{1}{1} = 1,\ \ \cfrac{1}{1 + \cfrac{1}{6}} = \frac{6}{7},\ \ \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{2}}} = \frac{13}{15},\ \ \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{2 + \cfrac{1}{6}}}} = \frac{84}{97},\ \ \frac{181}{209}, \frac{1170}{1351}, \ldots .$$

What to do with continued fractions

The **convergents** of $\frac{\sqrt{3}}{2}$ are:

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$$\left|\frac{181}{209}-\frac{\sqrt{3}}{2}\right|<\frac{1}{209\cdot 1351}=\frac{1}{282359}<\left|0.866-\frac{\sqrt{3}}{2}\right|$$

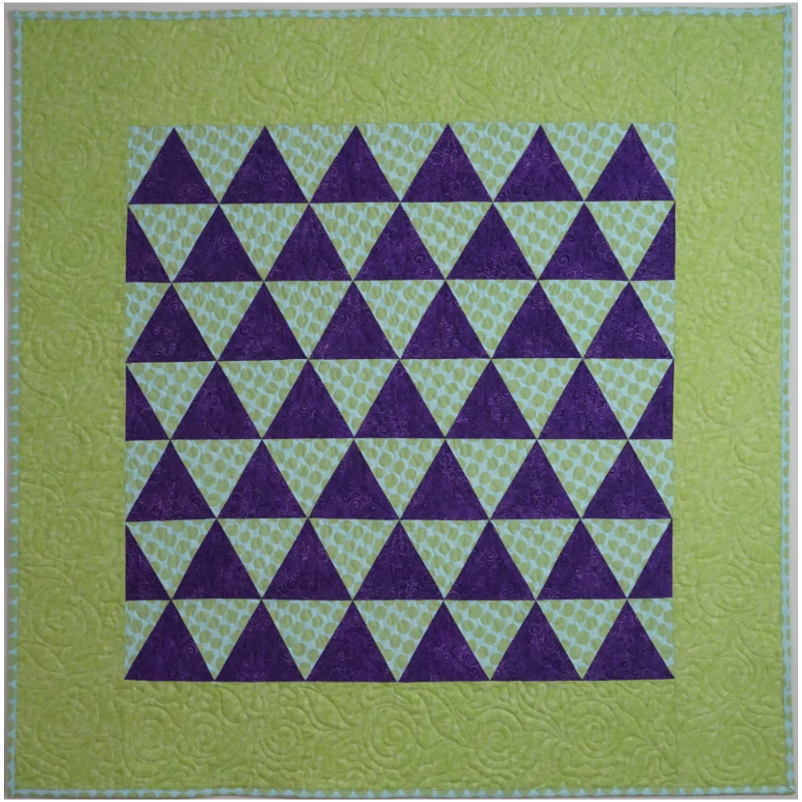
$$\left|\frac{13}{15}-\frac{\sqrt{3}}{2}\right|<\frac{1}{15\cdot 97}=\frac{1}{1455}<\left|0.87-\frac{\sqrt{3}}{2}\right|$$

$$\left|\frac{6}{7}-\frac{\sqrt{3}}{2}\right|<\frac{1}{7\cdot 15}=\frac{1}{105}<\left|0.9-\frac{\sqrt{3}}{2}\right|$$

A general fact:
For all positive real numbers x and for all positive integers b, d and non-negative integers a, c, if $d < b$, $\gcd(a, b) = 1$, and if $\frac{a}{b}$ is a convergent of x, then

$$\left|\frac{a}{b} - x\right| \leq \left|\frac{c}{d} - x\right|.$$

In other words, $\frac{a}{b}$ is a better approximation of x than $\frac{c}{d}$.



Adding borders with continued fractions

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The continued fraction for this ratio is $\dfrac{67}{49.5} = 1 + \dfrac{1}{2 + \frac{1}{1+\frac{1}{4+\frac{1}{1+\frac{1}{5}}}}}$.

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I used $\frac{23}{17}$ for my quilt.

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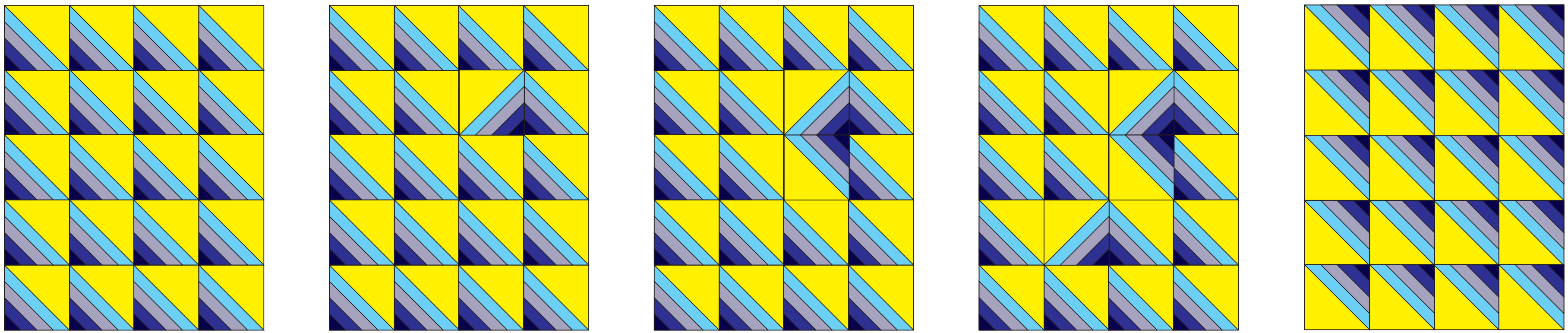
Possible arrangements for a 4 by 5 layout:



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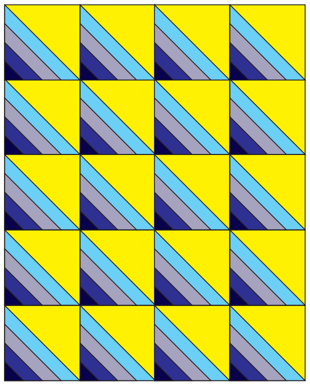 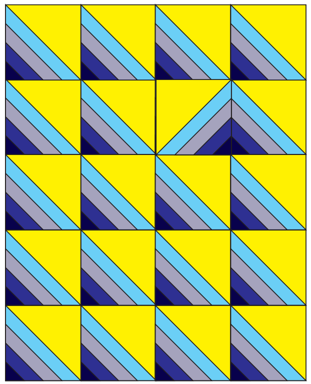 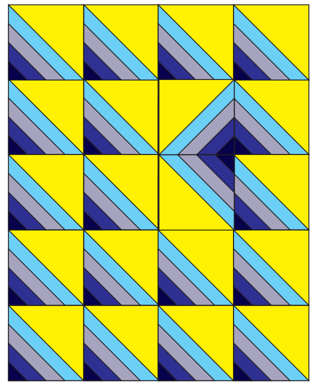 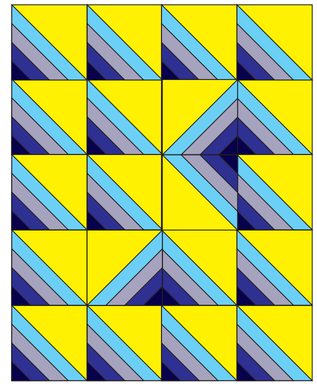 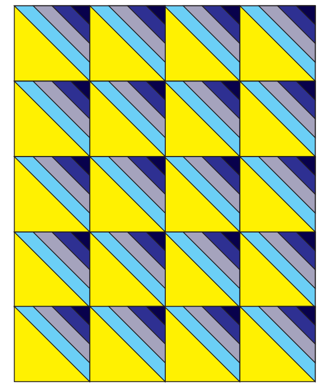

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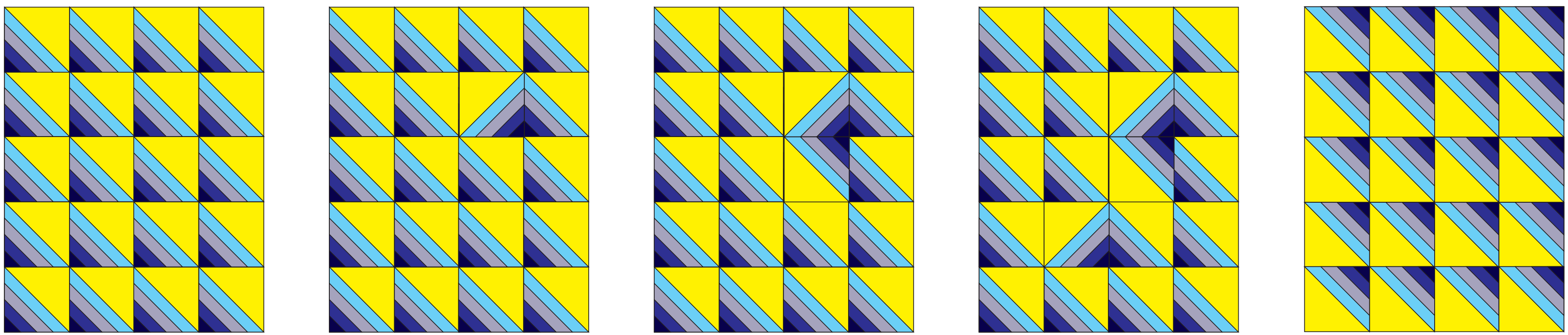

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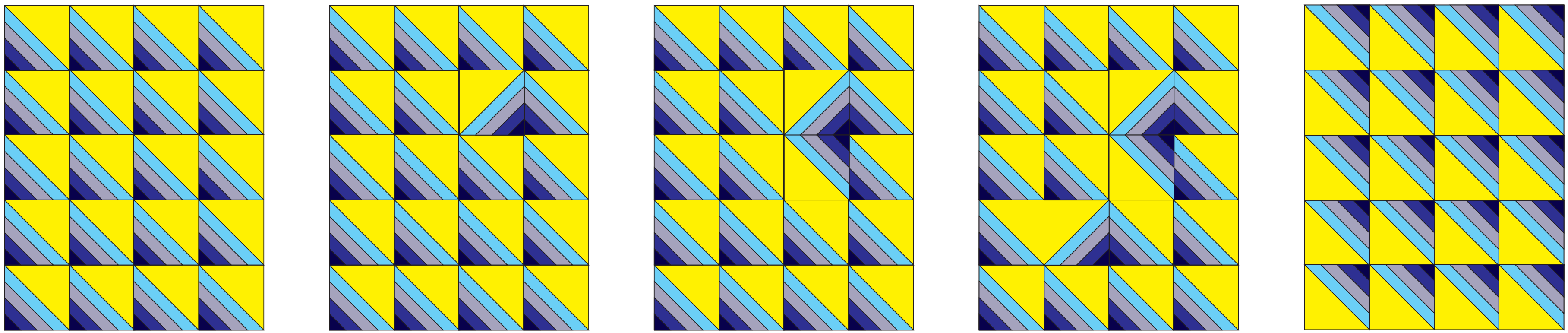

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How many different quilts create all these views?

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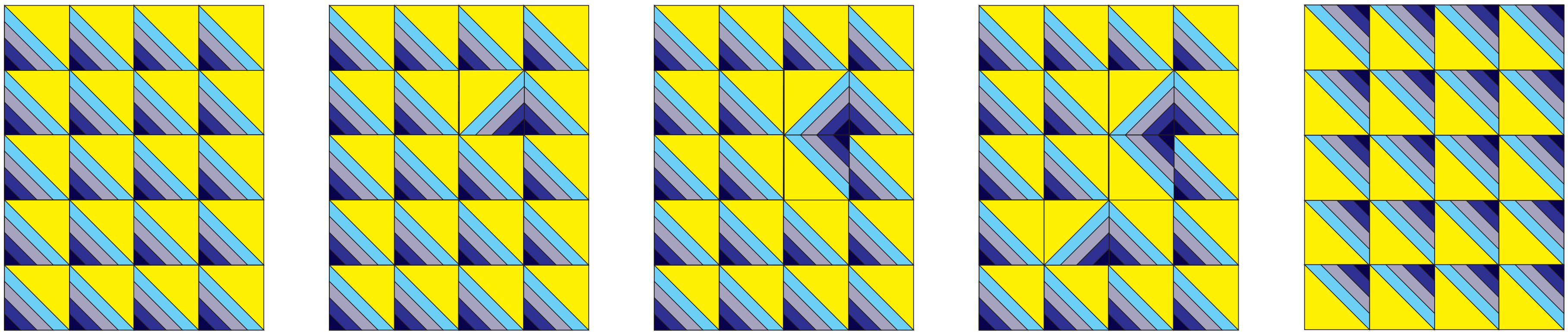

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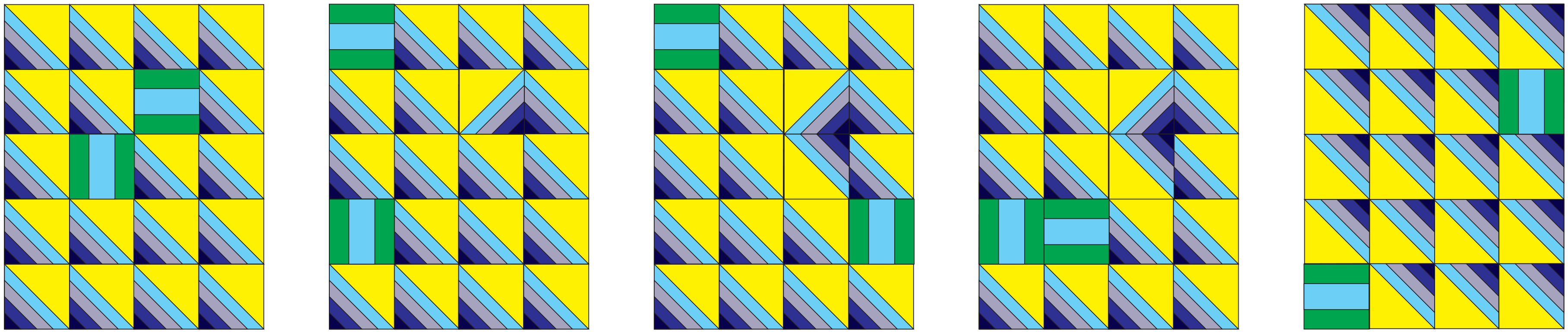
How many different quilts create all these views? $\frac{1}{2} \cdot 4^{20}$

What if we have 18 Roman stripes blocks with four different views and 2 rail fence blocks with two views?



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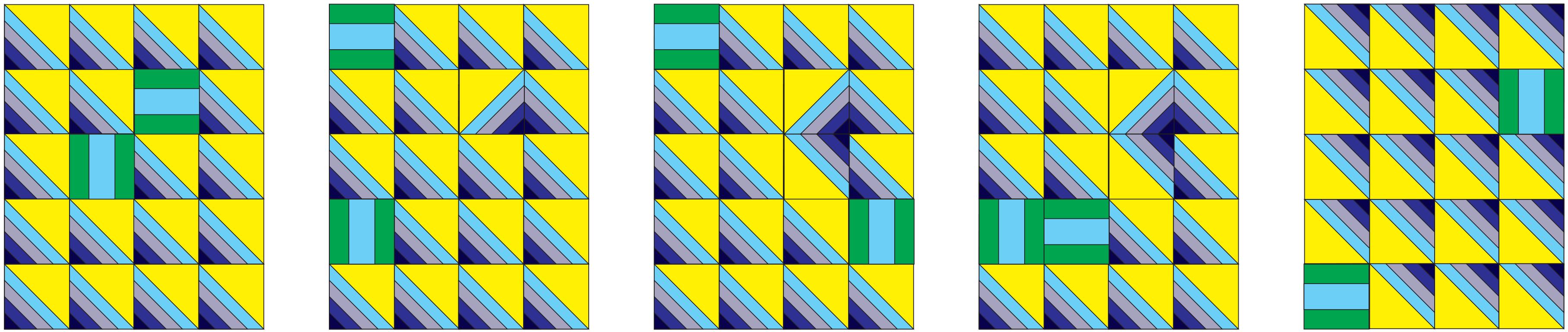

How many different views are there?

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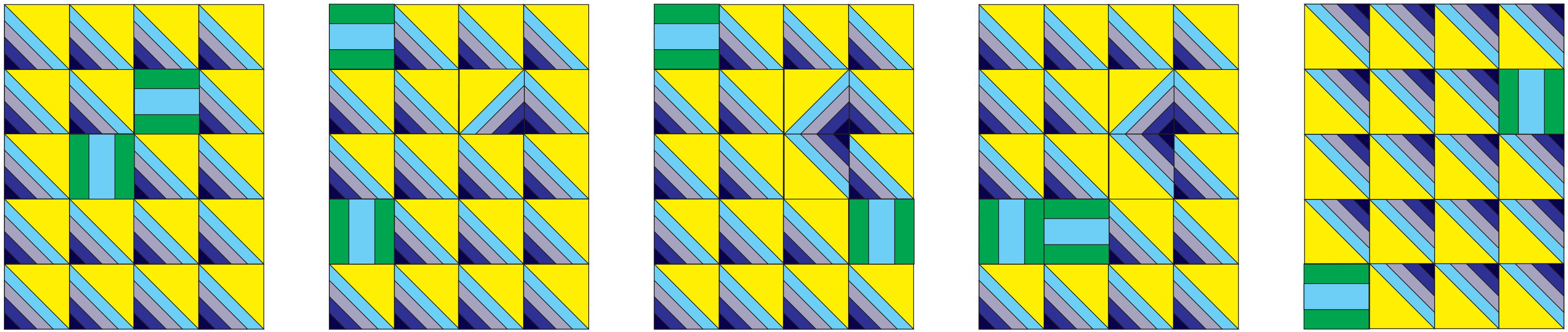

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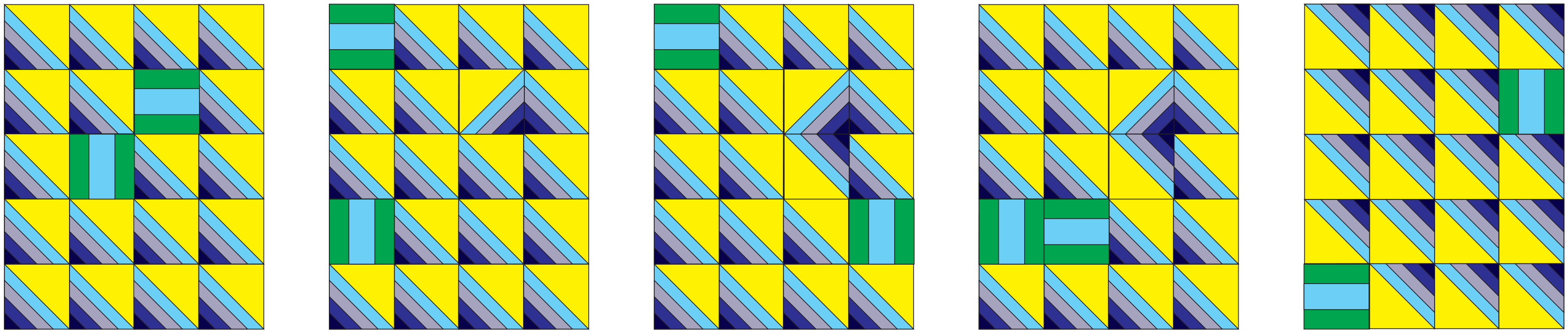

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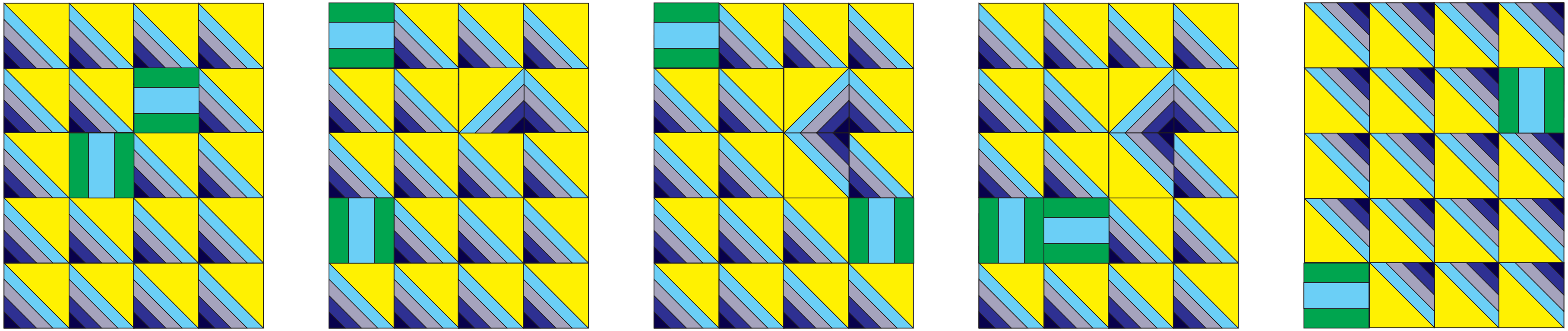
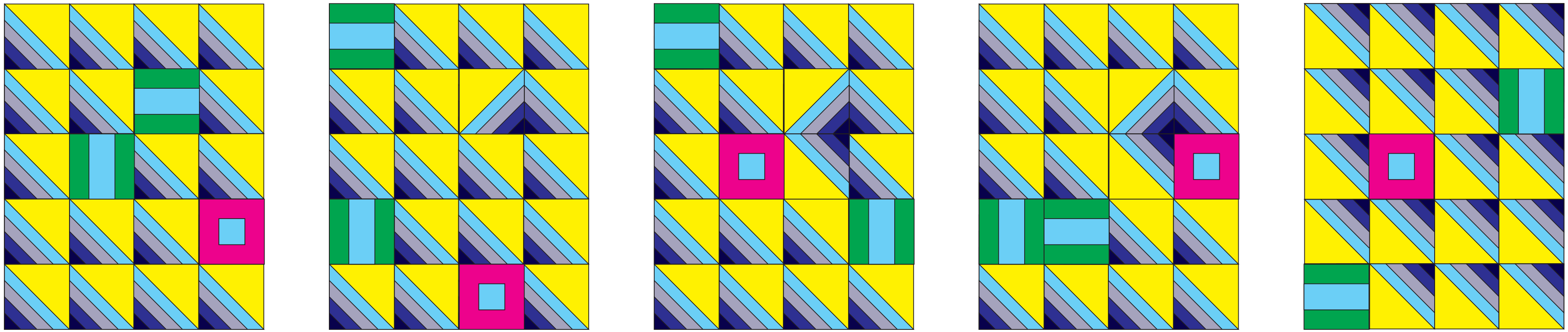


How many different views? $\binom{20}{2}2^2 \cdot 4^{18}$ (about $52 \cdot 10^{12}$)

How many different quilts create all these views? $\frac{1}{2}\binom{20}{2} \cdot 2^2 \cdot 4^{18}$

What if we have 17 Roman stripes blocks with four different views, 2 rail fence blocks with two views, and one block with only one view?



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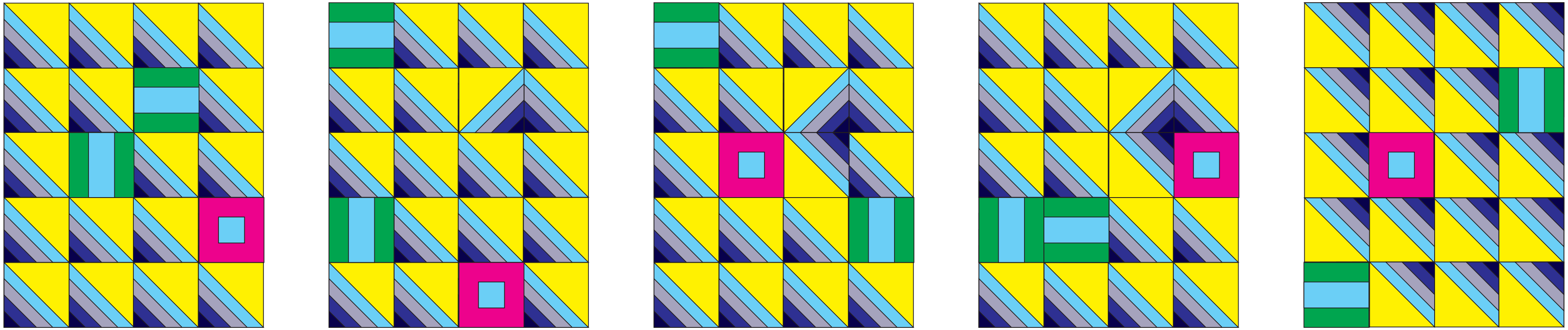

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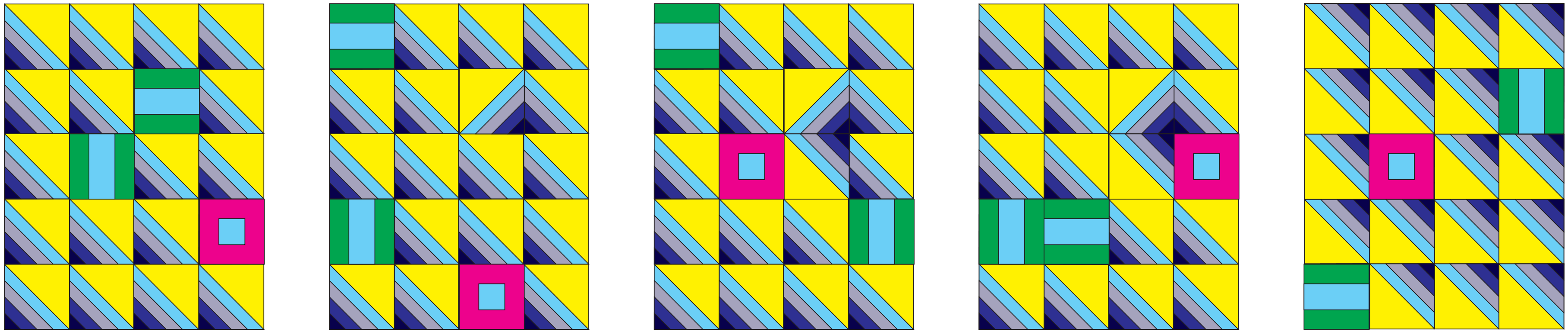

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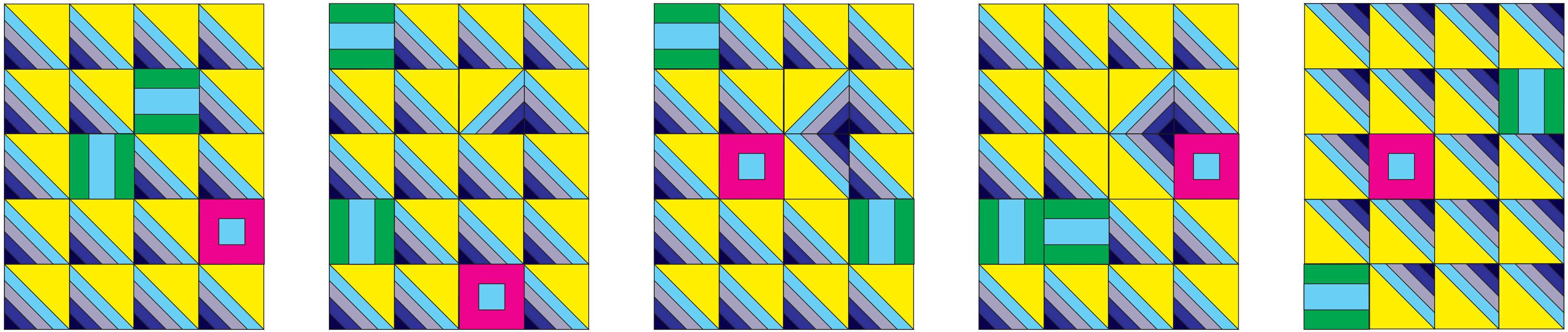

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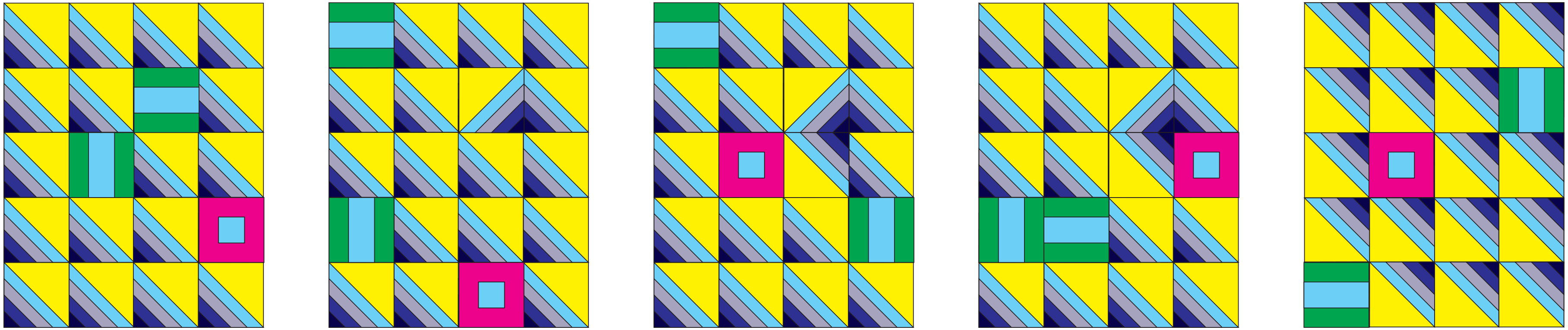

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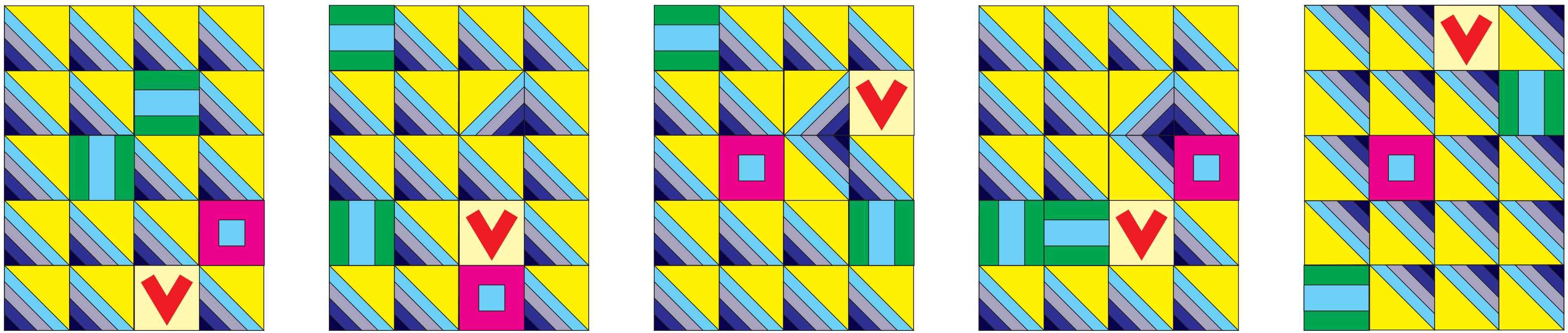
What if we have 16 Roman stripes blocks with four different views, 2 rail fence blocks with two views, one block with only one view and can be rotated, and one block that cannot be rotated (it has letter V)?



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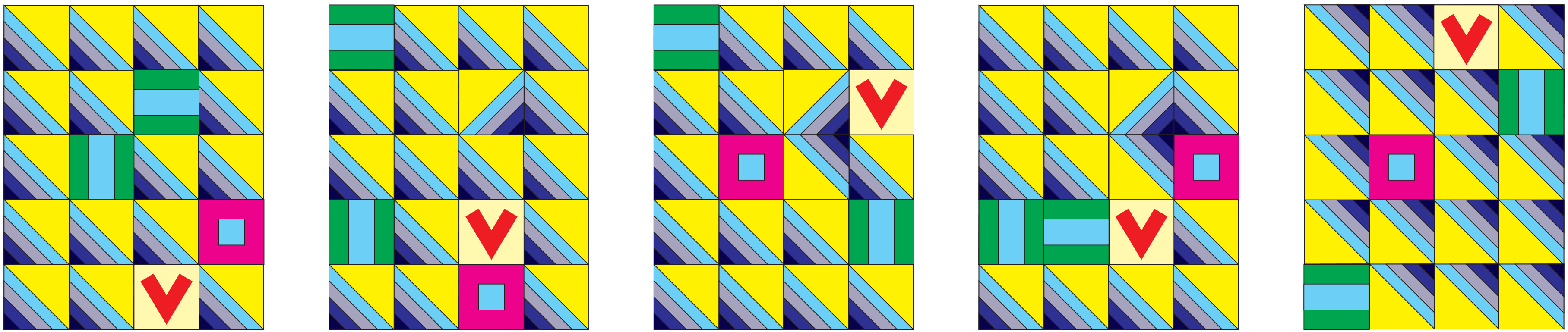

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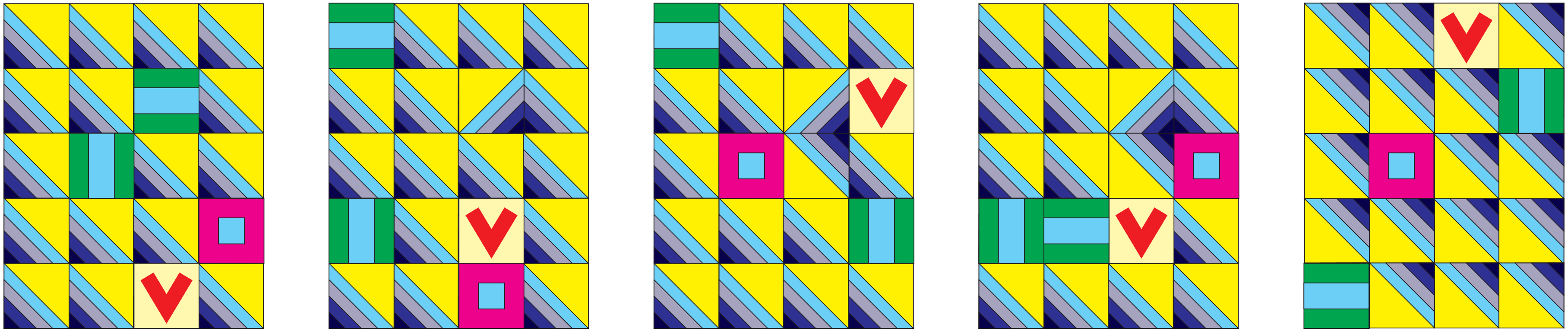

How many different views? $\binom{20}{2}\binom{18}{1}\binom{17}{1}2^2 \cdot 4^{16}$ (about 10^{15})

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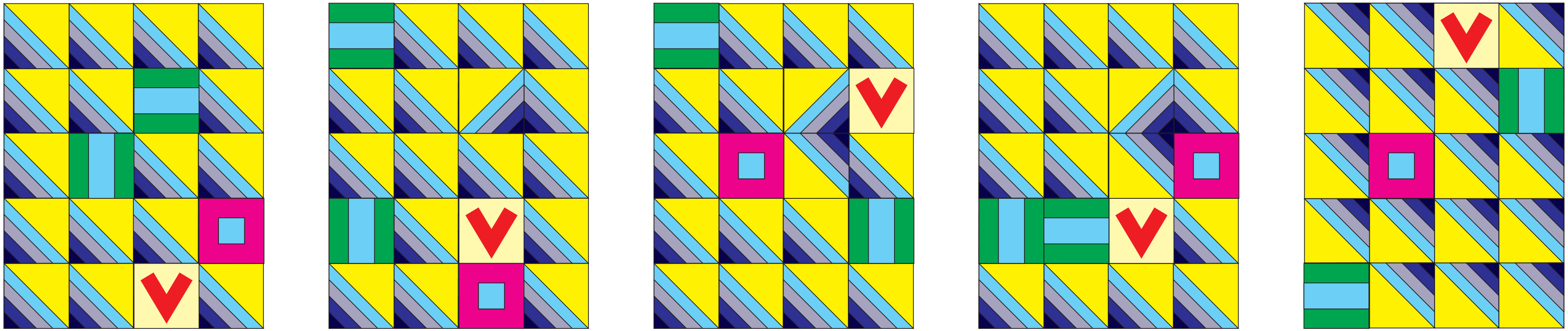

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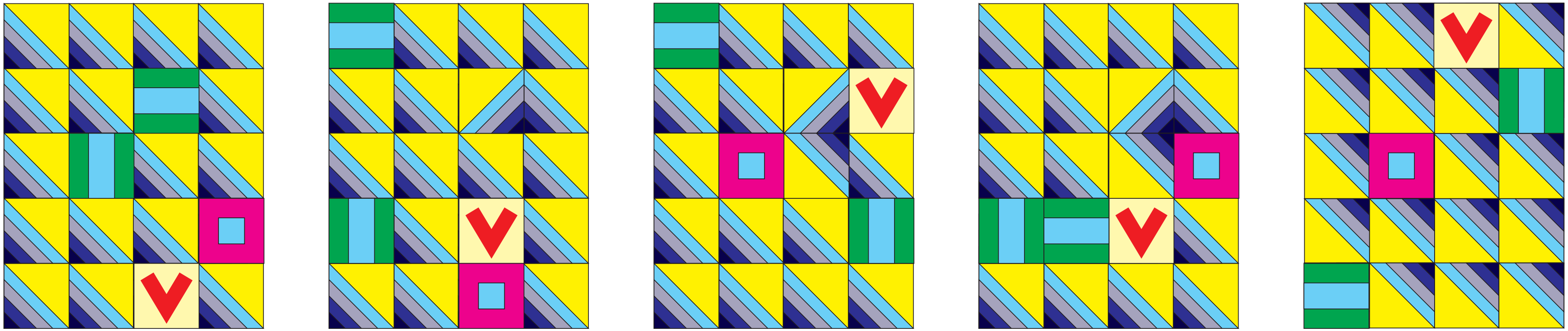

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How many different quilts create all these views? Same: $\binom{20}{2}\binom{18}{1}\binom{17}{1}2^2 \cdot 4^{16}$

What if we have 20 Roman stripes blocks with four different views, 20 rail fence blocks with two views, 20 blocks with only one view which can be rotated, and 20 blocks that cannot be rotated (say letter V). How many different quilt 4 by 5 quilt views can we make?

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Multiple choice:

A) 8^{20}

B) $\sum_{r_1=0}^{20}\left(\sum_{r_2=0}^{20-r_1}\left(\sum_{r_3=0}^{20-r_1-r_2}\binom{20}{r_1}\binom{20-r_1}{r_2}\binom{20-r_1-r_2}{r_3}4^{r_1}2^{r_2}\right)\right)$

C) about 10^{18}

What if we have 20 Roman stripes blocks with four different views, 20 rail fence blocks with two views, 20 blocks with only one view which can be rotated, and 20 blocks that cannot be rotated (say letter V). We want to make four 4 by 5 quilts. How many different quadruple-quilt views can we make?

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$$\sum_{r_1=0}^{20} \left(\sum_{r_2=0}^{20-r_1} \left(\sum_{r_3=0}^{20-r_1-r_2} \binom{80}{r_1} \binom{80-r_1}{r_2} \binom{80-r_1-r_2}{r_3} 4^{r_1} 2^{r_2} \right) \right)$$

(about $14 \cdot 10^{33}$)