Tessellations

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Reed College, Portland, Oregon

MathPath, Lewis & Clark College, Portland, Oregon, 24 July 2018

What is a tessellation?

A **tiling** or a **tessellation** of the plane is a covering of the plane with various (closed and countably many) shapes and with no overlaps and no gaps (other than overlaps on the boundaries of the shapes).

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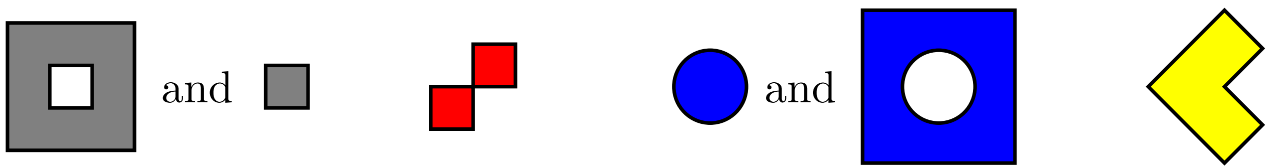
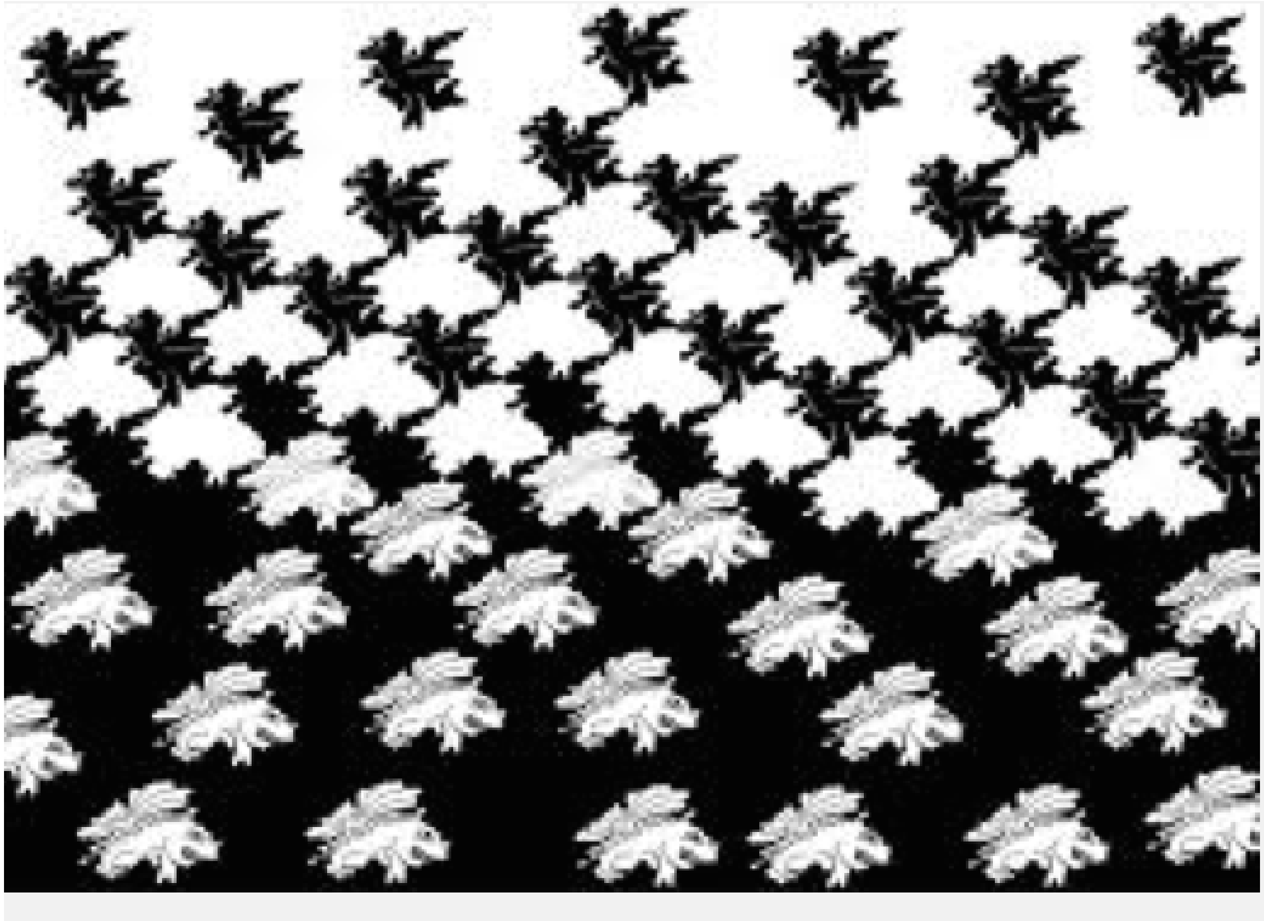
Some possible tiles:



M. C. Escher



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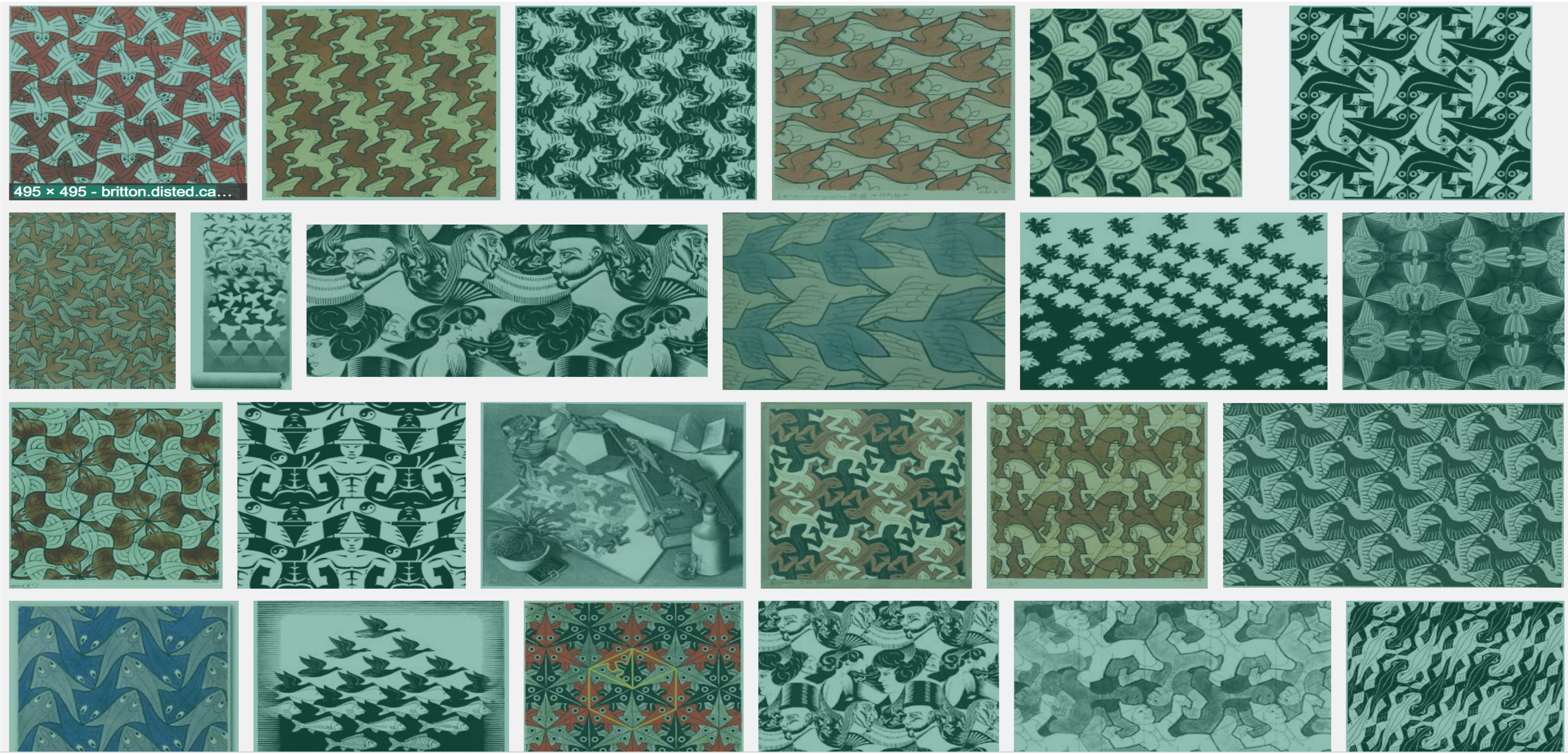

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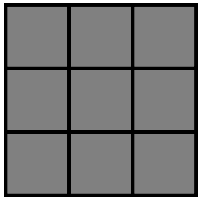
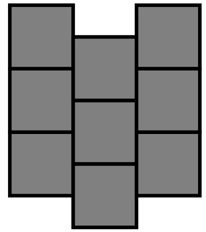
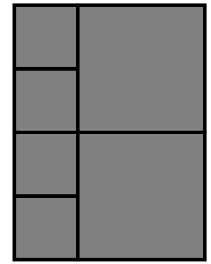
Some possible tessellations with regular polygons:

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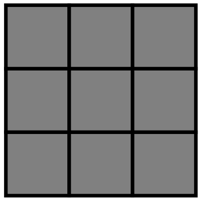
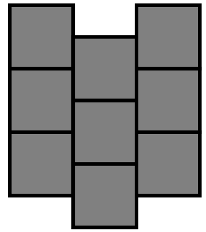
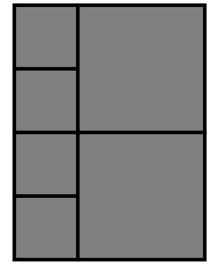
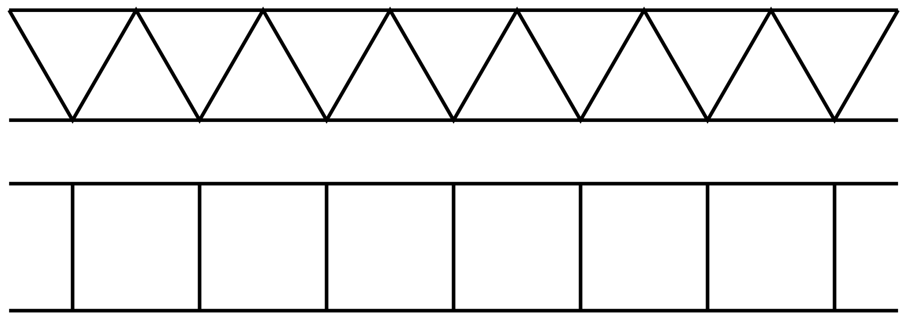
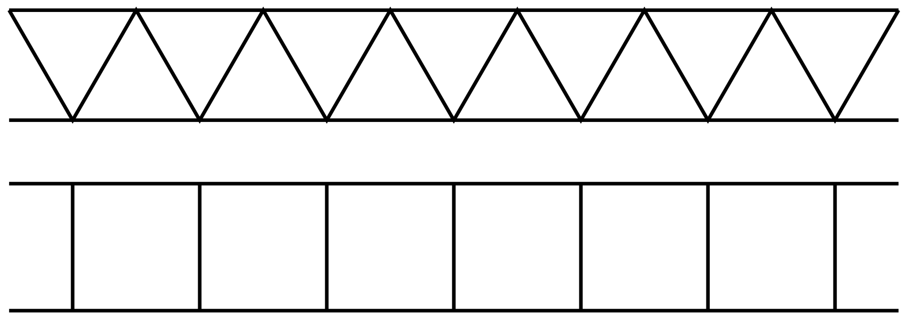




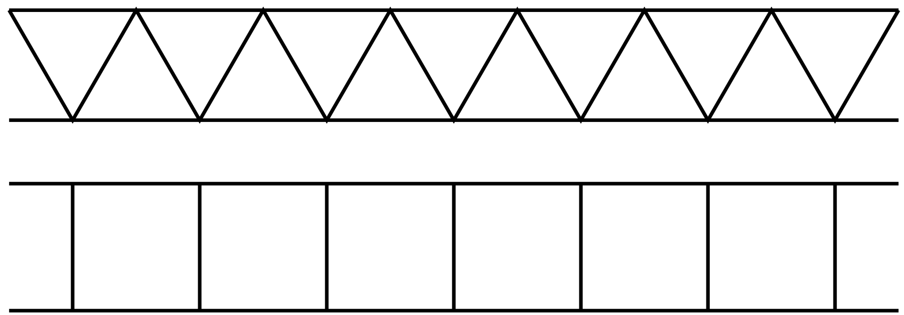


For today allow only edge-to-edge tessellations (only the leftmost example).





Stack these rows, matching vertices, but vary the numbers of types of strips. You get uncountably many tessellations.



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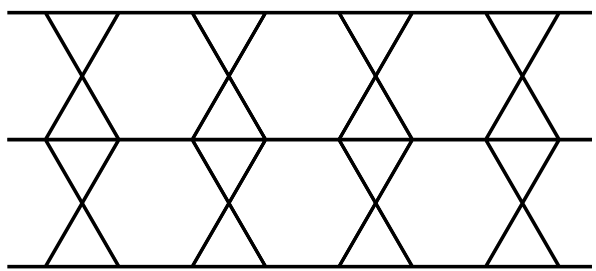
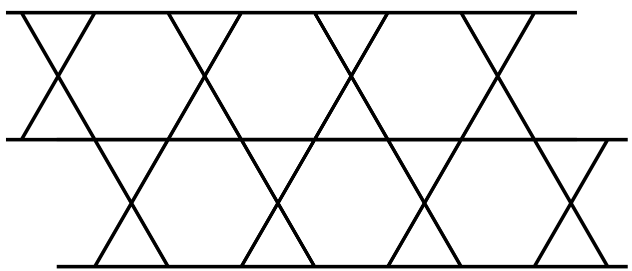
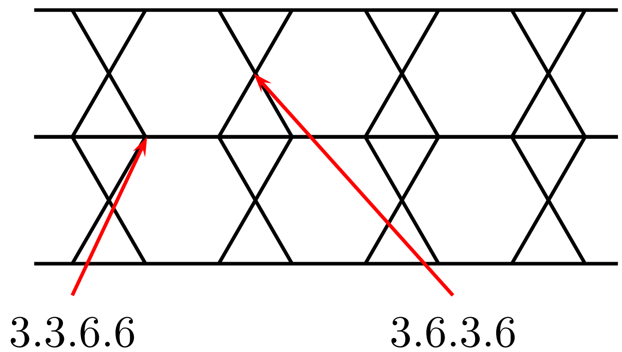
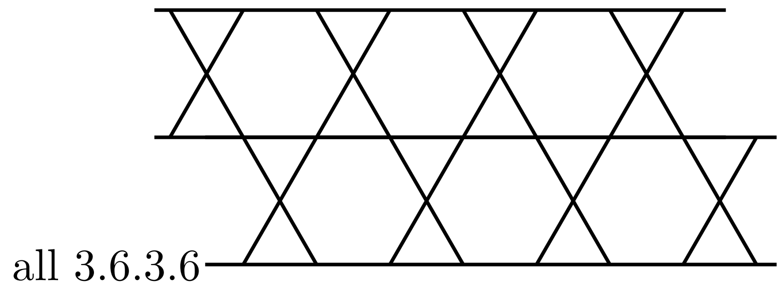
Or tessellate with identical rows of hexagons and triangles in uncountably many ways by stacking them in two different ways:





Type of a vertex:





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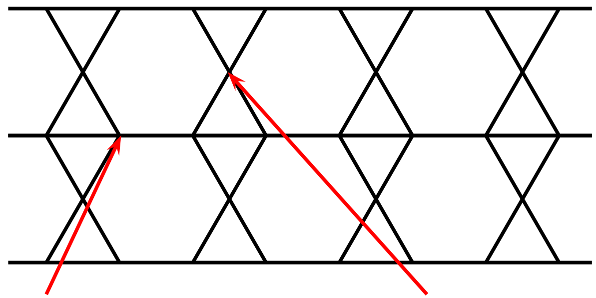

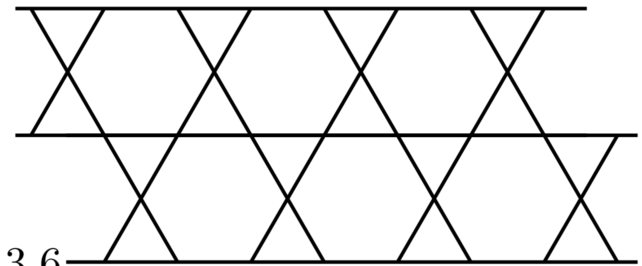

For now we restrict attention to edge-to-edge tessellations by regular polygons in which each vertex has the same type.

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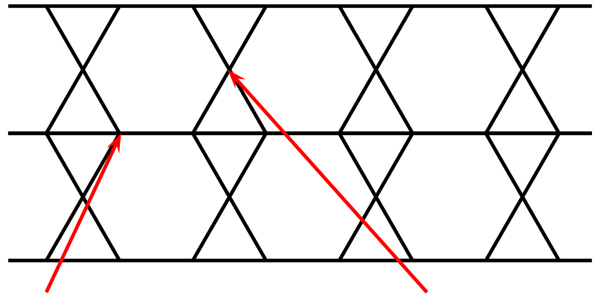
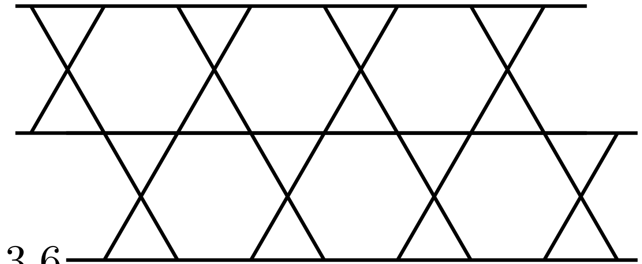


all 3.6.3.6



For now we restrict attention to edge-to-edge tessellations by regular polygons in which each vertex has the same type.

Such tessellations are called **semiregular** tessellations, or also **homogeneous** tessellations, **Archimedean** tessellations, **1-uniform** tessellations.

$\boxed{\textbf{Determining all semiregular tessellations}}$

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Some examples:

(1) Type 4.4.4.4 (all squares)
(2) Type 3.3.3.3.3.3 (all triangles)
(3) Type 6.6.6 (all hexagons)
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There are others. How do we find them all?

Theorem. A regular polygon with n sides (n-gon) has interior angles equal $180 - \frac{360}{n}$ in degrees.

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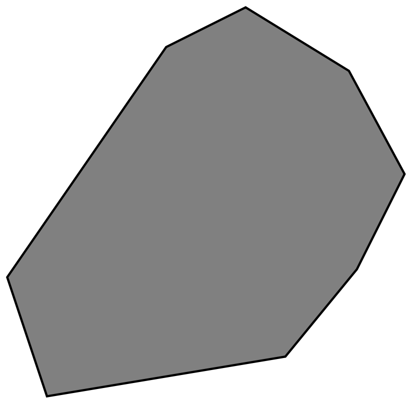

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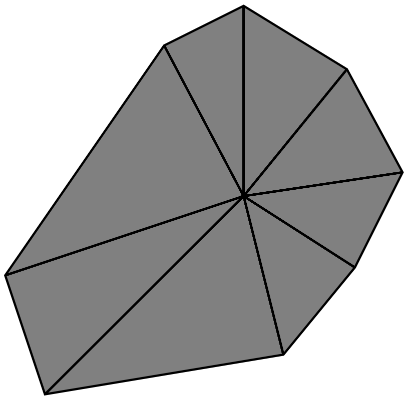

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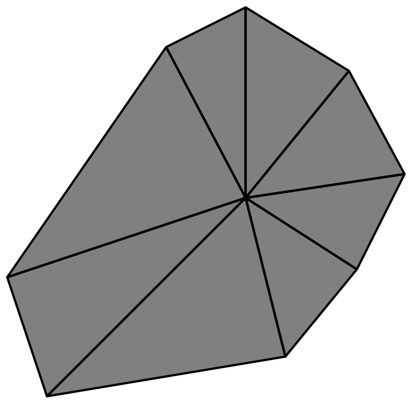

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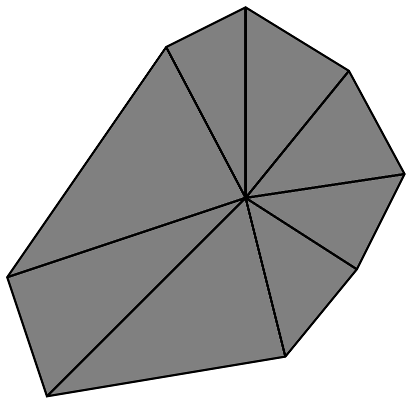

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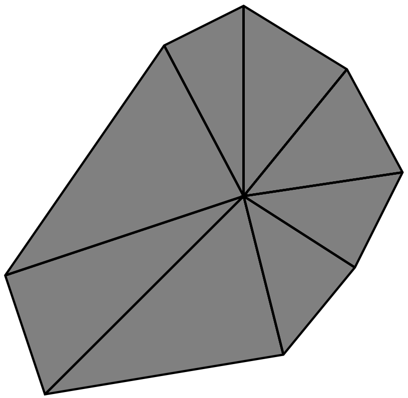

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Thus an interior angle in a regular n-gon measures $\frac{n-2}{n}180°$.

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Thus we get:
Theorem. In order for a regular n_1-gon, n_2-gon, ..., and an n_r-gon to meet at a vertex without overlaps and without gaps, it is necessary and sufficient that

$$\frac{n_1-2}{n_1}180° + \frac{n_2-2}{n_2}180° + \cdots + \frac{n_r-2}{n_r}180° = 360°.$$

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so that

$$\frac{r}{3}=r-r\frac{2}{3}\leq 2,$$

which means that $r\leq 6$.

So, we have to find all integer solutions $n_1, n_2, \ldots, n_r \geq 3$ of

$$r - 2 = \frac{2}{n_1} + \frac{2}{n_2} + \cdots + \frac{2}{n_r}$$

for $r = 3$, $r = 4$, $r = 5$ and $r = 6$.

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If $n_1 \leq n_2 \leq n_3$, then $n_1 \leq 6$.

If $n_1 = 3$, then we need to find n_2 and n_3 with $\frac{1}{6} = \frac{1}{n_2} + \frac{1}{n_3}$.

Solving $\frac{1}{6} = \frac{1}{n_2} + \frac{1}{n_3}$ (or $n_3 = 6 + \frac{36}{n_2 - 6}$):

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All solutions (with $r = 3$, $n_1 = 3 \leq n_2 \leq n_3$):
$n_2 = 7$, $n_3 = 42$,
$n_2 = 8$, $n_3 = 24$,
$n_2 = 9$, $n_3 = 18$,
$n_2 = 10$, $n_3 = 15$,
$n_2 = 12$, $n_3 = 12$.

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Still need all solutions to $r - 2 = \dfrac{2}{n_1} + \dfrac{2}{n_2} + \cdots + \dfrac{2}{n_r}$
for $r = 3$, $r = 4$, $r = 5$ and $r = 6$.

The complete list of integer solutions $n_1, \ldots, n_r \geq 3$ for the solutions of

$$\frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \cdots + \frac{n_r - 2}{n_r} = 2,$$

if we ignore the order of the n_i, is:

(1) $3,3,3,3,3,3$
(2) $3,3,3,3,6$
(3) $3,3,3,4,4$
(4) $3,3,4,12$
(5) $3,3,6,6$
(6) $3,4,4,6$
(7) $3,7,42$
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(9) $3,9,18$
(10) $3,10,15$
(11) $3,12,12$
(12) $4,4,4,4$
(13) $4,5,20$
(14) $4,6,12$
(15) $4,8,8$
(16) $5,5,10$
(17) $6,6,6$

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Warning: not all of these are realizable semiregular tessellations;

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Warning: not all of these are realizable semiregular tessellations; solution $\mathbf{3,3,3,4,4}$ yields two semiregular tessellations.

$\boxed{\textbf{Eliminate some options}}$

The vertex configuration $3, 3, 4, 12$ on the list could yield vertex types 3.3.4.12 or 3.4.3.12. Neither is possible. I show how to eliminate 3.3.4.12.

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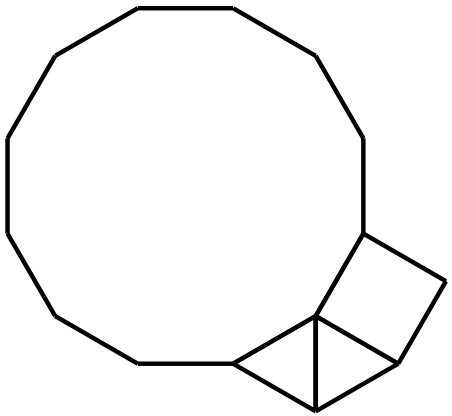

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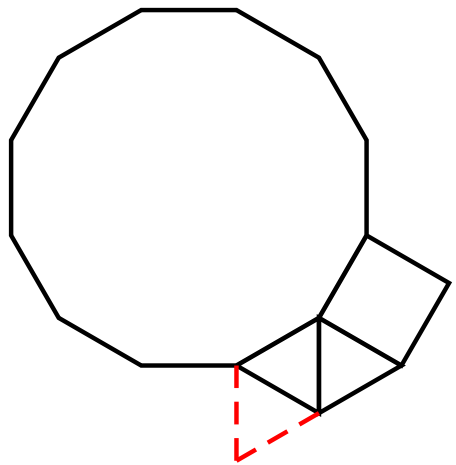
Forced to continue with a triangle:



but a triple triangle vertex is not allowed for $3, 3, 4, 12$.

Theorem. There are 11 semi-regular tessellations up to translations, rotations and reflections:

(1) 3.3.3.3.3.3	(5) 3.4.6.4	(9) 4.6.12
(2) 3.3.3.3.6	(6) 3.6.3.6	(10) 4.8.8
(3) 3.3.3.4.4	(7) 3.12.12	(11) 6.6.6
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(4) 3.3.4.3.4	(8) 4.4.4.4	

Compare to all solutions from before:

(1) $3, 3, 3, 3, 3, 3$	(7) $3, 7, 42$	(13) $4, 5, 20$
(2) $3, 3, 3, 3, 6$	(8) $3, 8, 24$	(14) $4, 6, 12$
(3) $3, 3, 3, 4, 4$	(9) $3, 9, 18$	(15) $4, 8, 8$
(4) $3, 3, 4, 12$	(10) $3, 10, 15$	(16) $5, 5, 10$
(5) $3, 3, 6, 6$	(11) $3, 12, 12$	(17) $6, 6, 6$
(6) $3, 4, 4, 6$	(12) $4, 4, 4, 4$